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Essays in Microeconomic Theory

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Abstract

This thesis investigates how introducing bounded rationality to a range of theoretical microeconomic situations can help explain empirical results.

Biased Screening and Coarse Information: A decision maker must choose one of two projects. Each project has an identically independently distributed value which is observed by an agent, but only a coarse signal can be communicated to the decision maker. Despite the symmetric setup, agents optimally partition the possible realisations of value differently to minimise the error cost in the decision stage. This is generalised to a screening process, where projects are grouped exogenously and different thresholds can be used to assess projects in different groups. A project in a minority group has a lower probability of being chosen, an insight which is related to the literature on categories and discrimination.

Endogenous Analogy Classes: An Analogy Equilibrium (Jehiel, 2005) involves players bundling nodes at which their opponent moves into analogy classes. The robustness of analogy classes is examined when players form their analogy classes endogenously, so are more likely to form analogies over nodes in which their opponent's behaviour is similar and when suboptimal actions would not prove very costly. These ideas are applied in a generalised Centipede game. Pure strategy equilibria involving passing may survive refinement in some long games, but mixed strategy equilibria can be dramatically more robust. In the most cooperative, robust equilibrium players pass for many nodes and mix at the end of the game.

Common Value Multi-Unit Auctions: The implications of using different multi-unit auction mechanisms are investigated when bidders have multi-unit demand and a common valuation V^m for the mth object they win. Revenue equivalence holds across many auction formats when bidders have constant or increasing marginal utility for additional units. When bidders have diminishing marginal utility however, an *almost common-values* problem means that inefficient first-price auctions are expected to raise the most revenue. Bounded rationality is important in explaining empirics and reinforces some of the theoretical results.

Disclaimer: I, Michael Lockhart Armitage, confirm that the work presented is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis. Parts 1 and 2 are new, whereas Part 3 is an improved and extended adaptation of a thesis which was previously submitted for my MPhil qualification. The differences are discussed in detail at the start of Part 3 on page 113.

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Chapter 1

General Introduction

This thesis investigates how introducing bounded rationality to a range of theoretical microeconomic situations can help explain empirical results and is divided into three parts. This first part introduces bounded rationality into a decision problem, by considering how a decision maker might optimally screen projects to choose one that maximises expected utility. The second investigates how bounded rationality might lead to players forming analogies about how their opponents will move, which could overcome the *finite horizon paradox* that arises in a generalised Centipede game. The third part develops a theoretical model of common value auctions in which bidders have multi-unit demand. Bounded rationality helps to explain empirical and experimental studies (multi-unit auctions are often complex) as well as reinforcing some of the theoretical results.

The first part of the thesis investigates how a decision maker might optimally screen projects to maximise expected utility. The screening heuristic is initially motivated as a coarse communication mechanism, where separate agents have private information which they summarise into simple recommendations for the decision maker. Screening is best motivated early in a search process, where communication, processing or information acquisition constraints generate coarse information and it is not yet optimal to use a more accurate, but costly, procedure involving sequential search or pairwise comparisons. Optimisation is motivated by learning as the screening problem is repeated over time. Despite the symmetry of the situation, it can be shown that it is optimal to introduce bias (or asymmetry) into the screening process and this result can be generalised to the case of partially asymmetric screening (or communication). In this case the decision maker can identify which group a project is in using an economically irrelevant characteristic, and assesses projects in different groups using different thresholds. In the optimal partially asymmetric screen, projects in the majority group face a higher threshold but are chosen in preference to projects in the minority group in the decision stage. Overall, the probability of a project in a minority group being chosen is lower than if it were a member of the majority group. These insights are related to some models from the economic theory of discrimination, where the use of biased screening tests is typical, as well as being used to explain recent empirical results on résumé screening.

An Analogy Equilibrium (Jehiel, 2005) involves players bundling nodes at which their opponent moves into analogy classes; players form beliefs consistent with the opponent's *average* behaviour in each analogy class and respond optimally to these beliefs. The second part of this thesis seeks to refine this approach, motivated by the idea that players form their analogy classes *endogenously* after observing past histories of the game. It is argued that players are less likely to form analogy classes over nodes in which the opponent's behaviour is very different, and that they form analogies more carefully when suboptimal actions could prove very costly. These motivations lead to two approaches to refine analogy classes, requiring that they be *robust*. Firstly, refinement could restrict the variation of behaviour permitted within an analogy class, which is based on the consistency between a player's analogy-based expectations and the beliefs he would hold if the analogy class were as fine as possible.[1] An alternative approach is to measure the suboptimality of a player's actions resulting from an analogy class. It is shown that these approaches lead to similar restrictions on behavioural strategies when they are applied formally to a class of timing games with complete and perfect information, such as Rosenthal's (1981) Centipede game. Generally pure strategy analogy-based expectations equilibria involving passing will survive refinement in long enough games, but mixed strategy equilibria can be dramatically more robust. This leads to an intuitive solution to the finite horizon problem: an equilibrium consists of players passing for a given number of nodes and then mixing towards the end of the game.

This third part of the thesis explores the implications for efficiency and expected revenue of using different multi-unit auction mechanisms when bidders have multi-unit demand and a common valuation V^m for the mth object they win. Bounded rationality helps to explain some related empirical and experimental results (multi-unit auctions are often complex) as well as reinforcing some of the theoretical results developed. When bidders have constant or increasing marginal valuations, the discriminatory, Vickrey and uniform price auctions have an equilibrium in which bidders submit flat demand curves. Empirically, however, bidders submit downward sloping

[1] These equal the opponent's actual behavioural strategies.

demand curves, which can be explained by decreasing marginal valuations.[2] In this case, the discriminatory and Vickrey auctions decompose into asymmetric single-unit auctions, so insights from the analysis of asymmetric auctions can be applied. In fact, although the Vickrey and uniform price auctions are efficient, the asymmetry creates an implicit *almost common values problem* so they generate low revenue. These theoretical results are compared to some empirical results from experimental economics and treasury auctions. While some of the empirical results might be explained by introducing more units and bidders, others, such as overbidding on a first unit, can only be explained by introducing bounded rationality, which seems reasonable as the complexity and uncertainty of the situation make it harder to condition on winning and to calculate optimal bidding strategies. Even if most bidders are able to fully comprehend the multi-unit auction, the presence of a few smaller bidders who are boundedly rational may introduce supply uncertainty.

Therefore in every part bounded rationality is related to the way *categories* or *analogies* can influence players' beliefs. In the first part, screening is analysed using a model in which information acquisition constraints cause projects to be evaluated using coarse *categories*. In the second, players form endogenous *analogy*-based expectations of their opponent's behaviour. Finally, the third part shows that although introducing decreasing marginal valuations explains some empirical results, the *analogy* that the opponent does not base his actions on his private information can explain overbidding, and the *analogy* with the case of an even division of the units can reinforce some of the theoretical results on low bid equilibria in uniform price auctions.

[2] Even if payoffs are constant, decreasing marginal valuations could be motivated implicitly as the reduced form of reciprocity considerations or risk aversion.

Part I

Asymmetric Communication and Biased Screening

Chapter 2

Introduction

The following thesis investigates how a decision maker might optimally screen projects to maximise expected utility.[1] The screening heuristic is initially motivated as a coarse communication mechanism, where separate agents have private information which they summarise into simple recommendations for the decision maker. The private information takes the form of a project value - although these are identically independently distributed, the thesis will show that it is optimal for agents to use different thresholds to partition these distributions, so they communicate different information to help the decision maker choose a project.[2]

The screening heuristic is best motivated early in a search process, where communication or information acquisition constraints generate coarse information, and it is not yet optimal to use a more accurate (but costly) procedure involving sequential search or pairwise comparisons. Therefore there are three key elements to this model: information is coarse, communication (or screening) is simultaneous and given these limitations, the decision maker and agents learn to optimise over time.

Coarse information could arise from constraints on communication, memory, processing ability or information acquisition. In the model presented here, the true information is represented by a one-dimensional random variable such as *quality* or *productivity*.[3] Coarse information is represented by a simple, discrete signal, for example that the realisation is either *low* or *high*, or *sell*, *buy* or *hold*. Many papers in the economics of discrimination assume that workers either *pass* or *fail* a test or

[1]Except where specified, *screening* refers to a simultaneous testing procedure, rather than the usual economic definition of informed players selecting from a menu of contractual offers.

[2]The agents and decision maker have the same objective, avoiding any principal-agent problems.

[3]The simplifying assumption that the information relevant to the decision can be aggregated into a one-dimensional variable is common to all of the papers discussed here on both bounded rationality and the economics of discrimination.

interview. As the partitioning becomes infinitely fine the decision problem reduces to the unconstrained case, but consistent with information theory it is assumed that finer partitions are more costly to communicate. This definition encompasses models in the literatures of bounded rationality and the economics of discrimination. Often signals are modelled as binary random variables which are dependent on quality (or some other measure representing true information), meaning errors may arise in communication as there is a chance a high quality project could send a *low* signal.[4] In contrast, the information structure in this thesis follows that of Dow (1991) in which there is no error in signals. This structure is seen as a simplifying assumption rather than a fundamental one, as it clarifies the underlying intuition and makes the model more tractable. When motivated using information acquisition constraints, coarse signals seem most reasonable early in the search process, when the small chance of choosing any specific project means that the expected gain from a detailed inspection or accurate communication is lower.

The assumption that firms engage in *simultaneous screening* of applicants is common in papers on the economics of discrimination.[5] The model developed here most closely resembles Cornell and Welch (1996) in which one winner is selected from a fixed number of applicants. The benefit of using a sequential search is that even if information is coarse, partitions can be optimally conditioned on previous signals (see Dow, 1991). However, this may require complex processing or considerably more communication, for example if the decision maker issues instructions back to agents during the search.[6] When the model is motivated using information acquisition constraints, a screening process seems reasonable early in the search process. Later on, once there are fewer objects, greater returns to accuracy could mean it is optimal to change the procedure and treat signals sequentially. This approach is consistent with some psychological models of consumer search. In addition, Arrow, Pesotchinsky and Sobel (1981) demonstrate that using a simultaneous binary screening procedure is a surprisingly powerful method of finding the maximum of a sample, so it is reasonable that a screening procedure is optimal early in a search for some specifications of communication costs.

The assumptions that information is coarse and search occurs through a simultaneous screen are consistent with several important models on the economics of discrimination. The third element better relates to models of *bounded rationality*: the decision

[4]For example Calvert (1985), Cornell and Welch (1996) and Lundberg and Startz (1983).

[5]Examples include Phelps (1972), Lundberg and Startz (1983) and Aigner and Cain (1977).

[6]Investigating the costs of information processing, Radner (1993) argues that pairwise comparisons should mean that the current optimal project is reprocessed in each comparison.

maker learns to make optimal decisions and to have the agents communicate the most useful information, given the limitations imposed by the model. As with many models containing elements of bounded rationality, the optimal solution may be complex to calculate. However, optimisation is motivated by learning as the screening problem is repeated over time. Therefore the decision maker might behave *as if* the optimal procedure were calculated without actually deriving it. The assumption of screening can also be motivated in this way: the decision maker learns to form thresholds optimally over time, but actually conditioning optimally on current information requires a more complete understanding of the situation. As the model is motivated in the contexts of bounded rationality and the economics of discrimination, the results are discussed in relation to the relevant literature on each area at the end of Chapters Three and Five.

Chapter Three will show that to minimise the expected cost of an error, asymmetry (or *bias*) in the screening process is optimal when signals are coarse. Therefore the agents optimally use different thresholds to partition possible realisations of project utility, despite the problem being symmetric *a priori*. One intuition for the bias is that agents use different language to communicate to the decision maker. One reports whether project X is *terrible* or *not terrible* while the other reports whether Y is *great* or *not great*. An alternative motivation is that the decision maker chooses an optimistic agent and a pessimistic agent. The optimist reports that X is *good* even when the realisation is only around average, while the pessimist has stricter standards and only reports that project Y is *good* if it is a very high realisation.[7] Proposition 3.3 will demonstrate that introducing a small bias into a symmetric screen leads to a first order gain and only a second order loss, therefore the optimality of bias is expected to hold generally for a range of different information structures. This insight will be related to similar models containing elements of bounded rationality including Dow (1991), Meyer (1991) and Calvert (1985).

The model that will be presented in this thesis can be generalised in a number of ways. The decision structure could be expanded to allow more options, or to allow the decision maker to choose more than one project. The information (or communication) structure could be generalised to increase the number of partitions (either in total of for each project) or the number of different partitions which may be used. To illustrate this last point, two extreme cases are the symmetric case, where all project values are partitioned in the same way, and the fully asymmetric case, where every threshold is different. This requires complex calculation and for the decision maker to remember a different threshold for every project.

[7] The first motivation is similar to Dow (1991) while the second is closer to Calvert (1985).

Chapter Five will investigate the partially asymmetric case between these extremes, where the decision maker may use some different partitions but partitioning every signal differently is too complex. Instead, the decision maker can identify which group a project is in using an economically irrelevant characteristic, and uses the same threshold for all projects within a group. In the optimal partially asymmetric screen, projects in the majority group face a higher threshold but are chosen in preference to projects in the minority group in the decision stage. Overall, projects in the minority group have a smaller chance of being chosen *a priori*. These insights will be related to some models from the economic theory of discrimination, where the use of biased screening tests is typical. Not only does this analysis suggest that a biased screening process might be optimal, but it also implies that this could lead to discrimination against a *minority*. For example, the intuitions developed in this thesis extend to the model of Cornell and Welch (1996), meaning that discrimination would be optimal *even without* their assumption that the decision maker gets less accurate signals from the minority group.

The thesis will briefly consider some recent interesting empirical results on résumé screening. This is particularly relevant as both coarse signals and a screening process are best motivated early in the search procedure. Bertrand and Mullainathan (2004) study racial discrimination in the US labour market by submitting fictitious résumés which are randomly assigned "African American" or "White" sounding names. They find that White sounding names receive 50% more callbacks for interview and that callbacks are more responsive to résumé quality for Whites. The analysis presented in this thesis could explain both these observations without any exogenous difference between individuals other than that they are identified with either a minority or majority group.

Chapter 3

Asymmetric Communication With Two Options

This chapter aims to show that asymmetric communication is optimal when information is coarse. It demonstrates that bias arises in a very simple and symmetric case to provide intuitions that can be extended to more complex environments. A decision maker must choose between one of two projects, labelled *Project X* and *Project Y*. The VN-M utility of each is a random variable X or Y with realisations x and y respectively. X and Y are identically independently distributed and it is assumed for simplicity that they are uniformly distributed over $[0, 1]$. The decision maker is aware of the distribution of X and Y but not their realisations. The realisations are observed by different agents, who only communicate a simple, coarse signal to the decision maker. Specifically, the agent observing X divides the possible utility space into two intervals "*high*" and "*low*", denoted by X_L and X_H, and communicates which interval the realisation x is in. Likewise the agent who observes Y reports whether the realisation y falls in interval Y_L or Y_H. Y_L denotes both an interval and the signal sent to the decision maker, so the decision maker receives a signal profile $S = \{X_{.}\ ,\ Y_{.}\}$. In Figure 3.1 the signal profile received is the realisation $s = \{X_H\ ,\ Y_L\}$. The decision maker chooses the project which maximises expected utility conditional on the signal received.

As well as choosing the optimal project given a specific signal, the decision maker is able to control how the agents report information.[1] In this simple example this means choosing the bounds of the intervals before signals are realised. Signals are *cheap talk* and it is assumed that agents report the coarse intervals truthfully, either because their

[1] Choosing how agents report is equivalent to choosing between a continuum of agents who report differently, as in Calvert (1985).

incentives are aligned with those of the decision maker or because the decision maker is able to use *ex post* monitoring.[2] Although described as a single decision problem, it is assumed that this situation is faced regularly, so the decision maker and agents learn to form intervals optimally over time.

Table 3.1 explains the process:

Stage 1	Decision maker partitions valuation space For $X\{X_L\ ,\ X_H\} = \{[0,a)\ ,\ [a,1]\}$ For $Y\{Y_L\ ,\ Y_H\} = \{[0,b)\ ,\ [b,1]\}$
Stage 2	One agent observes realisation x and the other observes realisation y
Stage 3	The agents communicate the partitions containing x or y to the decision maker, e.g. $s = \{X_H\ ,\ Y_L\}$ in Figure 3.1
Stage 4	The decision maker chooses the project that maximises expected utility conditional on the signals received.

Table 3.1: Timing of the communication process

This is illustrated in Figure 3.1:

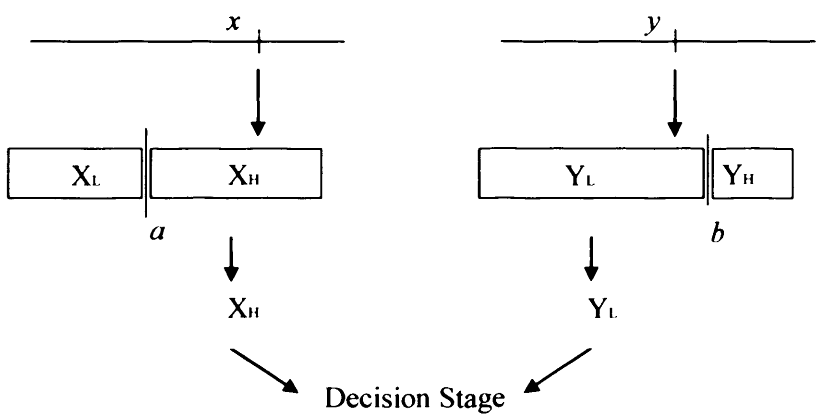


Figure 3.1: Timing of the communication process

In the decision stage, the decision maker maximises $E[V \mid s]$, the expected utility conditional on s, the realised signal profile. This equals $E[X \mid s]$ if Project X is chosen

[2]Therefore this abstracts from the literature following Crawford and Sobel (1982) investigating communication as a principal-agent problem when the interests of the agent and decision maker do not align.

and $E[Y \mid s]$ if Project Y is chosen. The optimal decisions conditional on s can be substituted back into the communication problem to determine the optimal partitions of X and Y. It is assumed throughout that the decision maker forms expectations using Bayes' law.

Two cases are now considered. In the first the agents communicate in the same way, so possible project realisations are partitioned identically for X and Y. In the second the signals may be partitioned into intervals differently, which is optimal even though the situation is symmetric *a priori.* In this thesis *asymmetric communication* and *biased screening* are used interchangeably to describe different motivations for the same model. In this sense the *bias* refers to the fact that identically distributed signals are communicated (or evaluated) differently and not that project space is partitioned evenly, nor that the decision stage is biased. For example, setting $a = b = \frac{3}{4}$ and choosing Y if both realisations are *high* or both *low* is considered *symmetric communication* or *unbiased partitioning.* The reason for this definition of bias is twofold: firstly, a test would not be considered biased just because the pass mark differs from 50%.[3] Secondly, if Y is always chosen when the decision maker is indifferent between X and Y this might constitute *decision bias* but not *screening bias.*

3.1 Symmetric Communication

In this case the same partitions are used to communicate information about both project utility values, so both random variables, X and Y, are partitioned into intervals $\{[0,a)\ ,\ [a,1]\}$. As the signals for a single project are mutually exclusive, the expected utility $E[V] = \sum_S E[V \mid s]\Pr[s]$ can be composed from the probabilities and conditional expectations shown in Table 3.2.

s	$E[X \mid s]$	$E[Y \mid s]$	Choice	$E[V \mid s]$	$\Pr[s]$
$\{X_L\ ,\ Y_L\}$	$\frac{a}{2}$	$\frac{a}{2}$	Y	$\frac{a}{2}$	a^2
$\{X_L\ ,\ Y_H\}$	$\frac{a}{2}$	$\frac{1+a}{2}$	Y	$\frac{1+a}{2}$	$a(1-a)$
$\{X_H\ ,\ Y_L\}$	$\frac{1+a}{2}$	$\frac{a}{2}$	X	$\frac{1+a}{2}$	$a(1-a)$
$\{X_H\ ,\ Y_H\}$	$\frac{1+a}{2}$	$\frac{1+a}{2}$	Y	$\frac{1+a}{2}$	$(1-a)^2$

Table 3.2: Probabilities and conditional expected utility under symmetric communication

If a specific signal leaves the decision maker indifferent between the projects, this means $E[X \mid s] = E[Y \mid s]$. In this case expected utility is the same whether a decision rule chooses X, Y or randomises between them. It is assumed for comparison with the asymmetric case (in Proposition 3.3) that an indifferent decision maker chooses Y,

[3]Therefore this definition is very different to the definition of bias in Calvert (1985).

but as the labelling of the projects is arbitrary at this stage, this is without loss of generality.

Proposition 3.1 *When the decision maker chooses between two options, optimal symmetric communication means dividing possible realisations of project utility into intervals*[4] $\{X_L\ ,\ X_H\} = \{Y_L\ ,\ Y_H\} = \{[0, a)\ ,\ [a, 1]\}$ *where* $a = \frac{1}{2}$, *to give* $E[V] = \frac{5}{8}$.

Proof. As the signals for a single project are mutually exclusive, every combination of such signals is mutually exclusive, so expected utility $E[V] = \sum_S E[V \mid s] \Pr[s]$. Substituting the probabilities and conditional expectations shown in Table 3.2 gives $E[V] = \frac{a}{2} + \frac{1}{2}\left[1 - a^2\right]$, which has a maximum at $a = \frac{1}{2}$ because $\frac{\partial E[V]}{\partial a} = \frac{1}{2} - a$ and $\frac{\partial^2 E[V]}{\partial a^2} = -1$. Substituting this into expected utility gives a maximum $E[V] = \frac{5}{8}$. ∎

This solution is illustrated in Figure 3.2 where possible realisations of X and Y are illustrated on the axes. The signals received conditional on the realisations are labelled in bold. Project Y is chosen in the shaded region and X in the unshaded region.

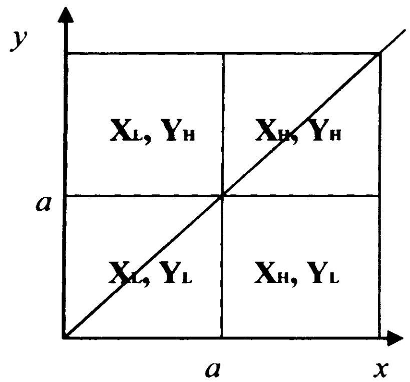


Figure 3.2: Optimal symmetric communication

In Figure 3.2 the 45° line shows the full information optimal choice of the decision maker, were he to observe the true realisations of X and Y. Above this line $y > x$ and so project Y is the true optimal choice, while below it $x > y$ and choosing X gives a higher utility. If the decision maker were able to observe realisations directly then the expected utility would be $E[V] = \frac{2}{3}$. Changing a affects the expected utility indirectly, by changing the usefulness of the information provided to the decision maker.[5]

[4]Dow (1991) shows that when when the expected payoff depends only on probabilities and conditional expectations, intervals are optimal. He gives an intuitive example that when x is divided into an *extreme* partition, $X_E = [0\ ,\ \frac{1}{4}) \cup [\frac{3}{4}\ ,\ 1]$, and a *moderate* partition $X_M = [\frac{1}{4}\ ,\ \frac{3}{4})$. This is totally uninformative as $E[X \mid X_E] = E[X \mid X_M] = \frac{1}{2}$. The use of intervals is more informative as it increases the difference between conditional expectations. Appendix A1 sketches a proof that intervals are optimal in the model presented in this chapter.

[5]In the case where $a = \frac{1}{2}$ one of the signals is redundant and provides no extra value. To see this observe that in Figure 3.2 the decision maker is indifferent in realisation X_L, Y_L, so the realised value

Proposition 3.2 shows that maximising $E[V]$ is equivalent to minimising the expected cost of errors.

Proposition 3.2 *Direct maximisation of expected utility is equivalent to minimising the expected error cost in the decision stage.*

Proof. Conditional on a specific signal $\widetilde{s}$, the full information expected utility, $E[V^*]$, always chooses X if $X > Y$ and Y if $X < Y$. Therefore

$$E[V^* \mid \widetilde{s}] = E[X \mid X > Y\ ,\ \widetilde{s}]\Pr[X > Y \mid \widetilde{s}] + E[Y \mid X < Y\ ,\ \widetilde{s}]\Pr[X < Y \mid \widetilde{s}]$$

Assume in realisation $\widetilde{s}$, the decision maker chooses X so $E[V \mid \widetilde{s}] = E[X \mid \widetilde{s}]$. The expected error conditional on $\widetilde{s}$ is $E[\varepsilon \mid \widetilde{s}] = E[Y - X \mid Y > X, \widetilde{s}]\Pr[Y > X \mid \widetilde{s}]$ so

$$\begin{aligned} E[V \mid \widetilde{s}] &= E[X \mid X > Y\ ,\ \widetilde{s}]\Pr(X > Y \mid \widetilde{s}) + E[Y \mid X < Y\ ,\ \widetilde{s}]\Pr(X < Y \mid \widetilde{s}) \\ &\quad + E[X - Y \mid X < Y\ ,\ \widetilde{s}]\Pr(X < Y \mid \widetilde{s}) \\ &= E[V^* \mid \widetilde{s}] - E[\varepsilon \mid \widetilde{s}] \end{aligned}$$

The same decomposition can be carried out for the case if Y (or any other project) is chosen conditional on a signal s. As signals are mutually exclusive

$$E[V] = \sum_S E[V \mid s]\Pr[s] = \sum_S E[V^* \mid s]\Pr[s] - E[\varepsilon \mid s]\Pr[s] = E[V^*] - E[\varepsilon] \quad (3.1)$$

Therefore $MaxE[V] \iff MinE[\varepsilon]$ because the full information expected utility, $E[V^*] = \frac{N}{N+1} = \frac{2}{3}$, is a constant (the expected maximum of two uniform distributions). ■

Figure 3.3 illustrates the errors under symmetric communication when project Y is chosen when the decision maker is indifferent.

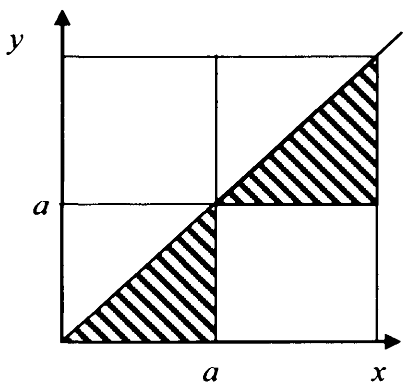


Figure 3.3: Errors under symmetric communication

is the same if X is chosen. Therefore the value is equal to when the decision maker chooses Y if Y_H is observed and X if Y_L is observed. Observing a signal of X confers no extra value in this case as the decision is the same either way.

The errors are illustrated as the shaded areas in which project Y is chosen but $x > y$ (below the 45° line). Appendix A.2 illustrates Proposition 3.2, deriving the optimal interval partitions by minimising expected error. In the symmetric case[6] illustrated in Figure 3.3 the expected error conditional on the realisation of a signal profile $\{X_L\ ,\ Y_L\}$ is $\frac{a}{6}$. The expected error conditional on $\{X_H\ ,\ Y_H\}$ is $\frac{1-a}{6}$, while there is no error under $\{X_L\ ,\ Y_H\}$ or $\{X_H\ ,\ Y_L\}$. Therefore expected error is given by Equation 3.2.

$$E[\varepsilon] = \sum_S E[\varepsilon \mid s] \Pr[s] = \frac{a}{6}a^2 + \frac{1-a}{6}(1-a)^2 = \frac{1}{6}\left[1 - 3a + 3a^2\right] \tag{3.2}$$

This is minimised when $\frac{dE[\varepsilon]}{da} = -\frac{1}{2} + a$, so optimally $a = \frac{1}{2}$, the optimal value of a derived using direct maximisation of $E[V]$ in Proposition 3.1. Substituting $a = \frac{1}{2}$ back into Equation 3.1 gives a minimum expected error of $\frac{1}{24}$, and as the full information expected value $E[V^*] = \frac{2}{3}$, Equation 3.2 gives $E[V] = E[V^*] - E[\varepsilon] = \frac{5}{8}$, confirming the solution in Proposition 3.1. This shows that the problem could be presented as a statistical decision problem, with a loss function equal to expected error (although in this case there is no error in *experiments* which are observed for independent projects).

3.2 Asymmetric Communication

This section analyses the case of *asymmetric communication* when agents may partition possible realisations of project value differently. It begins by demonstrating that the advantage of such bias holds very generally, then proceeds to solve specifically for the example when the underlying distribution of project values is uniform on $[0, 1]$.

Proposition 3.3 *Introducing bias to the case of symmetric communication increases expected value.*

Proof. From the symmetric case, assume that the threshold partitioning possible realisations of Y is increased by a small amount Δ. The second agent now reports whether y lies in the new intervals $\{Y_L\ ,\ Y_H\} = \{[0\ ,\ a+\Delta)\ ,\ [a+\Delta\ ,\ 1]\}$ while the first agent continues to partition X using $\{X_L\ ,\ X_H\} = \{[0, a)\ ,\ [a, 1]\}$. To conserve space, this gain is calculated by substituting $b = a + \Delta$ into the general solution for $E[V]$ in Proposition 3.4. This gives:

$$\begin{aligned} E[V(a\ ,\ a+\Delta)] - E[V(a\ ,\ a)] &= \frac{1}{2}\left[2a\Delta - \Delta^2 + a^2\Delta + a\Delta^2\right] \\ &= \frac{\Delta}{2}(1-a)^2 - \frac{\Delta^2}{2}(1-a) \end{aligned}$$

[6]This can be derived by putting $a = b$ in Table A2.1 in Appendix A2.

Therefore increasing the threshold on Y leads to first order gains and only second order losses. For any initial value of a, a small increase Δ increases $E[V]$. ■

Proposition 3.3 is illustrated in Figure 3.4. As in Figure 3.3, errors are represented by X being chosen below the 45^o line or Y being chosen above the 45^o line. Compared to the case of symmetric communication, the chance of an error when the realisation of X is *low* remains the same. The chance of both signals being *high* has reduced, although there is a new cost that sometimes in signal profile $[X_H\ ,\ Y_L]$, project X is chosen when in fact Y is optimal. These changes are summarised in the Figure 3.4.

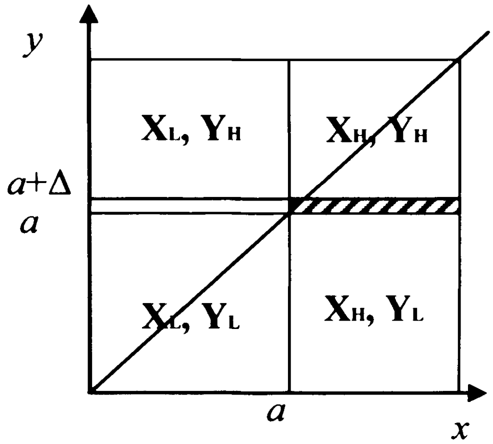


Figure 3.4: The gain from breaking symmetry

The first order gain $\frac{\Delta}{2}(1-a)^2$ is represented by the shaded area in Figure 3.4, where X is now chosen while in the symmetric case Y was chosen in error. The second order loss $\frac{\Delta^2}{2}(1-a)$ is represented by the solid black area, where now X is chosen while $Y > X$ was correctly chosen in the symmetric case. When project values are uniformly distributed, increasing a threshold to introduce bias increases $E[V]$ even when a is already suboptimally high. For example, if $a = 0.98$ then there is a gain from increasing one threshold even further to 0.99. As Proposition 3.4 is motivated using first order gains and second order losses, it holds very generally for unimodal continuous distributions of project value. These include cases in which sets are fuzzy, so there is a chance that realisations close to the threshold lead to the wrong signal, and a broad class of models where the coarse signal is positively correlated with underlying utility or quality. Cornell and Welch (1996) and Calvert (1985) are specific examples of such information structures.

3.2.1 Optimal Biased Partitioning in the Uniform Case

This section assumes that the decision maker and agents learn to partition the project valuation space optimally over time. Specifically the random variable X is partitioned into intervals $\{X_L\ ,\ X_H\} = \{[0,a)\ ,\ [a,1]\}$ while Y is partitioned into $\{Y_L\ ,\ Y_H\} = \{[0,b)\ ,\ [b,1]\}$. It is assumed that $a \leq b$.[7]

s	$E[X \mid s]$	$E[Y \mid s]$	Choice	Max $E[V \mid s]$	$\Pr[s]$
$\{X_L\ ,\ Y_L\}$	$\frac{a}{2}$	$\frac{b}{2}$	Y	$\frac{b}{2}$	ab
$\{X_L\ ,\ Y_H\}$	$\frac{a}{2}$	$\frac{1+b}{2}$	Y	$\frac{1+b}{2}$	$a(1-b)$
$\{X_H\ ,\ Y_L\}$	$\frac{1+a}{2}$	$\frac{b}{2}$	X	$\frac{1+a}{2}$	$(1-a)b$
$\{X_H\ ,\ Y_H\}$	$\frac{1+a}{2}$	$\frac{1+b}{2}$	Y	$\frac{1+b}{2}$	$(1-a)(1-b)$

Table 3.3: Probabilities and conditional expectations under asymmetric communication

As with symmetric communication, realisations of signals are mutually exclusive so expected utility, $E[V] = \sum_S E[V \mid s] \Pr[s]$, can be composed from the probabilities and conditional expectations given in Table 3.3.

Proposition 3.4 *When the decision maker chooses between two options, optimal asymmetric communication consists of dividing possible realisations of project utility for X into intervals $\{X_L\ ,\ X_H\} = \{[0,a)\ ,\ [a,1]\}$ and for Y into intervals $\{Y_L\ ,\ Y_H\} = \{[0,b)\ ,\ [b,1]\}$ where $a = \frac{1}{3}$ and $b = \frac{2}{3}$ giving $E[V] = \frac{35}{54}$.*

Proof. Substituting the probabilities and conditional expectations shown in Table 3.3 into $E[V] = \sum_S E[V \mid s] \Pr[s]$ gives $E[V(a\ ,\ b)] = \frac{1}{2}\left[1 + b - b^2 - a^2 b + ab^2\right]$. Therefore:

$$\begin{aligned}
\frac{\partial E[V(a\ ,\ b)]}{\partial a} &= \frac{b}{2}[b - 2a] \\
\frac{\partial E[V(a\ ,\ b)]}{\partial b} &= \frac{1}{2}(1-a)[1 + a - 2b]
\end{aligned}$$

so as $\frac{\partial^2 E[V]}{\partial a^2} = -b$ and $\frac{\partial^2 E[V]}{\partial b^2} = a - 1$, letting $a = \frac{1}{3}$ and $b = \frac{2}{3}$ gives a maximum *a priori* expected value of $E[V] = \frac{35}{54}$. ∎

This solution given in Proposition 3.4 is illustrated in Figure 3.5, where the realisations of project utilities, x and y, are shown on the axes and the signal profiles conditional on these realisations are labelled in bold. The realisations in which Y is chosen are shaded, while X is chosen in the unshaded region.

[7] There will be a second optimum where $a \geq b$ due to the symmetric nature of the problem.

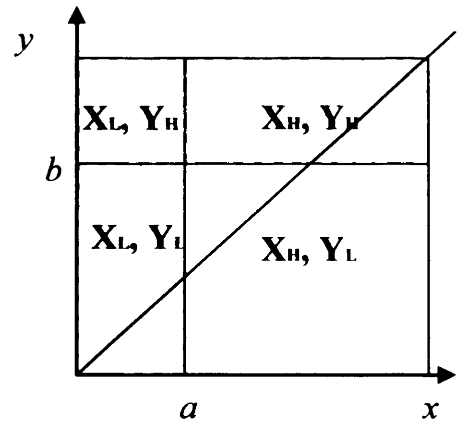
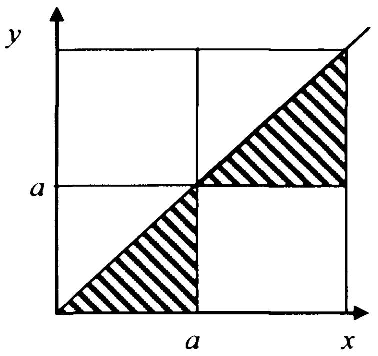


Figure 3.5: Optimal asymmetric communication

In Figure 3.5 the 45° line shows the full information optimal choice. Above this line $y > x$ so project Y is the optimal choice, while below it $x > y$ and choosing X gives greater utility. One intuition for the bias in this solution is that agents use different language to communicate to the decision maker. One reports whether project X is *terrible* or *not terrible* while the other reports whether Y is *great* or *not great*. An alternative motivation is that the decision maker chooses an optimistic agent and a pessimistic agent. The optimist reports that X is *good* even when the realisation is only around average, while the pessimist has stricter standards and only reports that project Y is *good* if it is a very high realisation.

3.2.2 Comparing the Optimal Unbiased and Optimal Biased Cases

Proposition 3.2 showed that maximising expected utility is equivalent to minimising the expected cost of errors. The minimisation of expected errors under asymmetric communication is derived in Appendix A.2 (as a demonstration of this) but expected error can be calculated using Equation 3.1 in Proposition 3.2 so $E[\varepsilon] = E[V^*] - E[V] = \frac{2}{3} - \frac{35}{54} = \frac{1}{54}$. When communication is symmetric, the expected error cost was calculated as $\frac{1}{24}$, so introducing asymmetry can reduce expected error by more than half. This is illustrated in Figures 3.6 and 3.7, where the shaded area represents an error conditional on realisations of X and Y in Propositions 3.1 and 3.4 respectively.



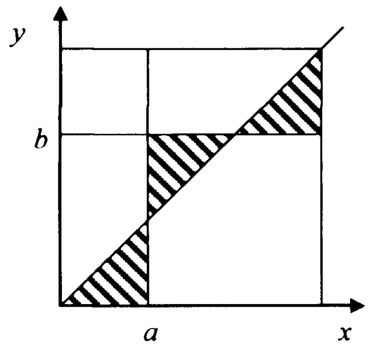


Figure 3.6: Errors under symmetric communication

Figure 3.7: Errors under asymmetric communication

Figures 3.6 and 3.7 illustrate that introducing bias reduces expected error in two ways: firstly, the overall probability of an error (represented as the total shaded area in these figures) is $\frac{1}{4}$ in the symmetric case compared to $\frac{1}{6}$ in the optimal asymmetric solution. Secondly, the cost of any error is $X - Y$ below the 45° line and $Y - X$ above it. This is represented by the vertical distance to the 45° line. As well as the probability of errors being smaller in Figure 3.7 than Figure 3.6, on average the errors which occur are closer to the 45° line and are therefore smaller in magnitude.

3.3 Discussion

The key insight of this chapter is that when signals are coarse it is optimal to introduce bias to the communication stage. It was shown that introducing a small bias leads to first order gains and second order losses, a general result which is expected to hold for all unimodal continuous distributions.[8]

The model of partitioning is very similar to that in Dow (1991), although in this case it is motivated where signals arrive *simultaneously* rather than *sequentially*. This leads to an intuition in common with Calvert (1985) who considers how a rational decision maker can optimally use imperfect advice.

Dow (1991) assumes that an individual's memory is limited, preventing him remembering the exact price of an item. Instead, an agent partitions the set of possible prices and remembers which partition the price is in. In a second round, the agent observes the exact price at a second shop. This is compared to what he expects the first price to be, conditional on it being within the memorised partition. The agent then purchases the item from the store with the lowest expected price.

[8]The optimal partitioning will, of course, depend on the specific distribution.

Meyer (1991) assumes that partitions represent limitations on communication within an organisation. The agent in this case observes a noisy signal of the amount produced by two workers in the previous period. Although able to make a *pairwise comparison* (which is not possible in the model presented in this thesis) the agent is only able to communicate the ordinal information, which worker produced most, to the decision maker.[9] Signals arrive sequentially and typically asymmetric evaluation criteria are optimal, as the decision maker tries to learn as much as possible about the difference in the underlying ability of the workers. The optimal final-period bias favours the leader.

In common with the model presented in this thesis, Calvert (1985) assumes signals arrive simultaneously and does not allow pairwise comparisons (despite a single agent observing both signals). Unlike this thesis, however, his model involves errors in communication.[10] Agents are represented by a single advisor who sends binary signals that each project is *good* or *bad*. There is an underlying distribution of project value over $U_i \sim U[0,1]$ and the probability of reporting a project as *good* is $u_i^{\alpha_i}$ while the probability of reporting it is *bad* is $1 - u_i^{\alpha_i}$. An unbiased source in his model is interpreted as $\alpha_1 = \alpha_2 = 1$, so for each single project the advisor is equally likely to report that it is *good* or *bad*. He shows that when $\alpha_1 = 1$, it is optimal to set $\alpha_2 = 3.45$, and argues that here bias is optimal because it increases the accuracy of high realisations at the cost of reducing accuracy for moderate valuations. However, there are two types of bias acting in Calvert's model. The first is *absolute bias*, as given the distribution he uses, reporting that a project is *good* or *bad* with equal probability (i.e. $\alpha_1 = \alpha_2 = 1$) is not optimal, and it is better to increase α_i for both projects. The second is *relative bias*, that it is optimal to set $\alpha_1 \neq \alpha_2$. Calvert's analysis bundles these effects as he breaks asymmetry from suboptimal case of $\alpha_1 = \alpha_2 = 1$ and maintains $\alpha_1 = 1$. In contrast, this thesis defines bias as $a \neq b$ and therefore only considers *relative bias*. By assuming communication takes place without errors, the simpler analysis allows the optimal level of bias (i.e. optimal asymmetric communication) to be determined[11] and it is possible to show that the bias relates to the underlying information structure and so holds generally for a range of different distributions. Calvert also shows that when the decision maker favours one of the projects *a priori*, then optimally he chooses an advisor who is biased in favour of it.

Another related approach is the economics of communication through codes. Wern-

[9] In contrast to this model, as pairwise comparisons are possible, bias would not be optimal in the one period case.

[10] In the sense that even if a project has a very bad realisation, there is a small probability that a *good* signal will be sent.

[11] In contrast, Calvert (1985) only solves for the optimal value of α_2 when $\alpha_1 = 1$.

erfelt (2004) argues that there are compatability advantages of agents communicating using the same code, it can make a large difference whether a language arises as the result of optimisation or an equilibrium of a team theory problem. In Wernerfelt's model, as in Radner and Marshack's (1972) analysis of team theory, every informed agent also acts, in contrast to this thesis in which only the uniformed decision maker chooses an action. In this regard the model presented in this chapter is more similar to models of decentralised processing[12] rather than decentralised decision making.

This thesis can be contrasted with Radner (1993). In common with much of the computer science literature, he focuses on finding a process which generates the optimal choice as quickly and cheaply (in terms of processing power) as possible. For example in Radner (1993), efficiency is motivated in the sense that the optimal solution cannot be reached without either longer delay or more processors. The question addressed in this thesis is how to minimise expected error given specific limitations in processing or communication, rather than how to minimise the degree of (costly) processing required to give the optimal solution. Arrow, Pesotchinsky and Sobel (1981) consider how to use binary partitions optimally to find the t largest observations in a sample size n. This is related to setting a in the unbias case, although their aim is to minimise either the number of rounds of questioning or maximise the probability that the search terminates within a given number of rounds. They demonstrate that using binary partitions is a surprisingly powerful method if the partitions are chosen optimally. In this sense their approach is similar to Radner (1993), as the question they address is how to find the optimal project as efficiently as possible, not how to get as good a project as possible for a given number of rounds of questioning.

[12] See Van Zandt (1999) for a survey on this literature.

Chapter 4

Generalising the Model

The model presented in Chapter Three can be generalised in a number of ways. The decision structure could be expanded to allow more options, or to allow the decision maker to choose more than one project. The information (or communication) structure could be generalised to increase the number of partitions (either in total or for each project) or the number of different partitions which may be used. To illustrate this last point, two extreme cases are the symmetric case, where all project values are partitioned in the same way, and the fully asymmetric case, where every threshold is different. This requires complex calculation and for the decision maker to remember a different threshold for every project. Chapter Five will investigate the case in between, where the same threshold is used for all projects in a group.

To provide benchmarks for the analysis in Chapter Five, this chapter characterises optimal communication in the symmetric and fully asymmetric cases with N projects. A conjecture is also made about the optimal spread of partitions when the decision maker could focus attention on a single project. When there are many projects the notation is altered slightly: the ith project is labelled X_i, since it would provide utility equal to a random variable X_i, with a realisation x_i. X_i is partitioned into intervals $\{[0, a_i) \ , \ [a_i, 1]\}$.

4.1 Symmetric Communication with N Projects

In this case N random variables X_i are partitioned into the same intervals $\{[0, a) \ , \ [a, 1]\}$. The simplest way to calculate the optimal value of a is to define H as the number of *high* realisations and condition expected value on its realisation, h. $E[V] = \sum_H E[V \mid h] \Pr[h]$ can be composed from the probabilities and conditional expectations shown in Table 4.1.

Number of *high* results h	Value $E[V_i \mid h]$	$Pr(h)$
$h \neq 0$	$\frac{1+a}{2}$	$1 - a^N$
$h = 0$	$\frac{a}{2}$	a^N

Table 4.1: Probabilities and conditional expectations conditional on h

Proposition 4.1 calculates the value of a which maximises $E[V]$.

Proposition 4.1 *When the decision maker chooses between N options, optimal symmetric communication divides the realisation of project valuations into intervals* $\{[0,a)\ ,\ [a,1]\}$ *where* $a = \left(\frac{1}{N}\right)^{\frac{1}{N-1}}$, *to give* $E[V] = \frac{1}{2}\left[1 + \frac{N-1}{N}\left(\frac{1}{N}\right)^{\frac{1}{N-1}}\right]$.

Proof. The realisations $h = 0$ and $h \neq 0$ are mutually exclusive so expected utility $E[V] = \sum\limits_{H} E[V \mid h] \Pr[h]$. Substituting the probabilities and conditional expectations shown in Table 4.1 gives $E[V] = \frac{1}{2}\left[1 + a - a^N\right]$, which has a maximum at $a^* = \left(\frac{1}{N}\right)^{\frac{1}{N-1}}$ because $\frac{dE[V]}{da} = \frac{1}{2}\left[1 - Na^{N-1}\right]$ and $\frac{\partial^2 E[V]}{\partial a^2} = -\frac{(N-1)Na^{N-2}}{2} < 0$ providing $N \geq 2$. Therefore the maximum $E[V] = \frac{1}{2}\left[1 + \frac{N-1}{N}\left(\frac{1}{N}\right)^{\frac{1}{N-1}}\right]$. ■

The optimal $a^* = \left(\frac{1}{N}\right)^{\frac{1}{N-1}}$ means a^* increases as N increases, such that as $N \to \infty$, $a^* \to 1$. This is intuitive when motivated as a screening mechanism, as the more options there are available the tougher the screening process should be. This relationship is illustrated in Figure 4.1 for $N \geq 2$.[1]

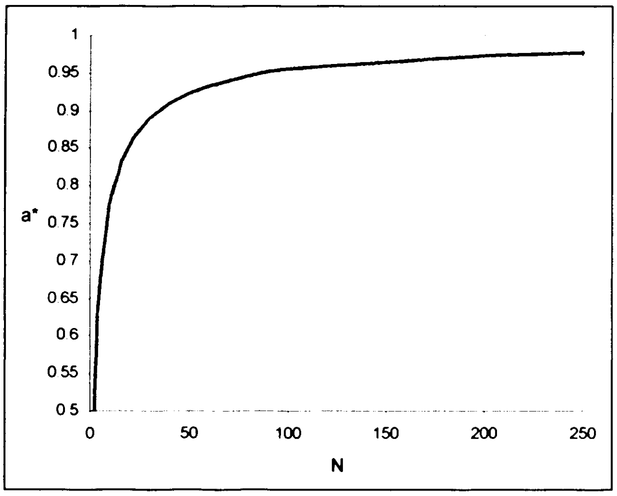


Figure 4.1: Optimal symmetric communication with N projects

When $N = 2$, optimally $a^* = \left(\frac{1}{2}\right)^{\frac{1}{1}} = \frac{1}{2}$, demonstrating that Proposition 4.1 is a generalisation of Proposition 3.1.

[1] Technically N is discrete.

Example 4.1 *When* $N = 6$, $a^* = \left(\frac{1}{6}\right)^{\frac{1}{5}} = 0.6988$. *This gives an expected value* $E[V] = 0.7912$ *meaning the expected error equals* 0.0660.

Using the motivation of a screening process rather than a communication problem, this symmetric threshold can be related to Simon's (1955) notion of satisficing, using a single aspiration level. Satisficing is proposed as an alternative to optimising behaviour, where an economic agent (in this case the decision maker) sequentially compares projects to a predefined aspiration level of utility,[2] and accepts the first which is above this level. In the model presented in this thesis, signals arrive simultaneously rather than sequentially.

4.2 Fully Asymmetric Communication with N Projects

In this case each project X_i is partitioned by the respective agent into different intervals indexed by a threshold a_i so $\{X_{iL}\ ,\ X_{iH}\} = \{[0, a_i)\ ,\ [a_i, 1]\}$. It is assumed that $a_1 \geq a_2 \geq ...a_{N-1} \geq a_N$ without loss of generality (the optimal partitioning is independent of the labelling of projects, which is arbitrary). $E[V]$ can be calculated from the probabilities and conditional expectations shown in Appendix A.3 to give Equation 4.1.

$$E[V] = \frac{1-a_1^2}{2} + a_1\frac{1-a_2^2}{2} + ... + [a_1a_2...a_{n-1}]\frac{1-a_n^2}{2} + [a_1a_2...a_{n-1}a_n]\frac{a_1}{2} \tag{4.1}$$

Proposition 4.2 calculates the values of a_i which maximise $E[V]$.

Proposition 4.2 *When the decision maker chooses between N options, optimal asymmetric communication divides possible realisations of project valuations into intervals* $\{[0, a_i)\ ,\ [a_i, 1]\}$ *where the thresholds,* a_i*, satisfy Equations* 4.2, 4.3 *and* 4.4.

Proof. Appendix A.3 derives Equation 4.1 and shows that when all a_i are set optimally they satisfy:

$$a_1 = \frac{1+a_2^2}{2} + \frac{1}{2}[a_1a_2...a_{n-1}a_n] \tag{4.2}$$

$$a_i = \frac{1+a_{i+1}^2}{2} \tag{4.3}$$

$$a_n = \frac{a_1}{2} \tag{4.4}$$

This maximises $E[V]$. Substituting the optimal solution into Equation 4.1 gives the maximum $E[V] = \frac{1}{2}\left[1 - a_1^2 + a_1\left(1 + a_2^2\right)\right]$. ■

[2] In this example, the threshold a could represent a constant aspiration level.

It is straightforward to verify that when $N = 2$, $a_2 = \frac{a_1}{2}$ and therefore $a_1 = \frac{1}{2}\left(1 + a_2^2 + [a_1 a_2]\right) = \frac{1}{2} + \frac{3a_1^2}{8} \implies a_1 = \frac{2}{3}$, $a_2 = \frac{1}{3}$. In this case $E[V] = \frac{1}{2}\left[1 - a_1^2 + a_1\left(1 + a_2^2\right)\right] = \frac{35}{54}$. This illustrates that Proposition 4.2 is a generalisation of Proposition 3.4.

Example 4.2 *Following Proposition 4.2, when $N = 6$ optimal asymmetric communication consists of each project X_i being partitioned into intervals of the form $\{[0, a_i)\ ,\ [a_i, 1]\}$ where the thresholds a_i satisfy Equations 4.2, 4.3 and 4.4. The values of a_i satisfying these equations are given in Table 4.2.*

a_1	a_2	a_3	a_4	a_5	a_6
0.8286	0.7632	0.7255	0.6716	0.5858	0.4143

Table 4.2: Optimal asymmetric thresholds when N=6

This partition structure is illustrated in Figure 4.2,[3] *which shows how a_i partition the possible realisations of utility for each project X_i by marking the optimal asymmetric thresholds with squares. Example 4.1 showed that in the optimal symmetric case $a =$ 0.6988, giving an expected error of 0.0659. This is illustrated in Figure 4.2 as a dashed line.*

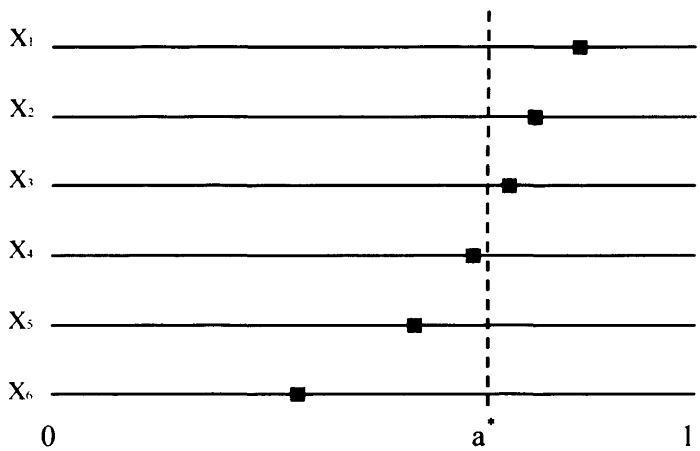


Figure 4.2: Optimal communication with N=6

The expected utility from the fully biased case is $E[V] = \frac{1}{2}\left(1 - a_1^2 + a_1\left(1 + a_2^2\right)\right) =$ 0.8123, so the expected error equals 0.0448. Therefore allowing fully asymmetric communication reduces the expected error by 32%.

Some intuition for the optimal values of a_i given in Proposition 4.2 can be gained by defining X_i^L as the probability that all $[X_1, X_2, ..., X_i]$ up to X_i are *low*. This is X_N^L if all realisations of X_i are *low*, and the decision maker optimally chooses X_1, because

[3] When $N > 2$ it is no longer possible to represent realisations in two dimensions so an alternative representation is used.

$E\left[X_i \mid X_i^L\right] = \frac{a_i}{2}$, so X_1 is optimal as $a_1 \geq a_2 \geq ...a_{N-1} \geq a_N$.[4] As $E\left[V \mid X_N^L\right] = \frac{a_1}{2}$ and $\frac{a_1}{2} = a_N$ (from Equation 4.4), $a_N = E\left[V \mid X_N^L\right]$. Therefore it follows from Equation 4.3 that for all $i > 1$:

$$\begin{aligned} a_i &= \frac{1}{2}\left(1 + a_{i+1}^2\right) = \frac{1 + a_{i+1}}{2}\left(1 - a_{i+1}\right) + a_{i+1}a_{i+1} \\ &= E\left[X_{i+1} \mid X_{i+1} > a_{i+1}\right]\left(1 - a_{i+1}\right) + E\left[V \mid X_{i+1}^L\right] a_{i+1} \\ &= E\left[V \mid X_i^L\right] \end{aligned}$$

Therefore an implication of Proposition 4.2 is that for all a_i when $i > 1$, optimally $a_i = E\left[V \mid X_i^L\right]$ so the threshold dividing intervals on signal X_i is set to equal the expected utility if all signals up to an including X_i have *low* realisations.

The optimal fully asymmetric case is similar to a sequential search, inspecting first X_1, then X_2 and continuing until a signal is *high*, such as the *house selling problem* of Simon (1955). An individual selling a house sets an acceptance price each day and receives an offer, the distribution of which is known. If the offer is above the acceptance price, it is accepted, otherwise it is rejected and the house is retained until the next day. The problem is how to optimally set the acceptance price each day. This problem is also discussed as a variant of the secretary problem in Moser (1956). In both cases there is an implicit stopping problem, with an optimal solution characterised by setting $a_N = 0$, $a_{N-1} = \frac{1}{2}$ and $a_i = \frac{1}{2}\left(1 + a_{i+1}^2\right)$. The difference between this and the solution derived in Proposition 4.2 is that in the latter, the decision maker can recall earlier applicants, so in the final turn chooses X_1. Although the model is very different, the recurrence relation is identical. This is because in both cases the decision maker is prevented from conditioning on the observed history, either because signals arrive simultaneously or because past applicants cannot be recalled. Under coarse information, even if the projects are screened sequentially in order of strictest to easiest cutoffs, Conjecture 4.1 implies it is not be optimal to recall earlier applicants.[5] The only case of recall is when all projects have *low* realisations and X_1 is chosen. Of course, any permutation of the optimal screening process can be reordered from the strictest to easiest cutoff to provide the intuition of recall. Simon (1955) argues that an individual without full information of the situation might act in a way approximating the optimal procedure without carrying out the optimisation itself.

[4] When motivated as a screening process, X_1 is the optimal choice if all projects fail the screen because it has failed the hardest test.

[5] In the house selling or secretary problems, it would be trivial to choose the best applicant if recall were permitted, as the decision maker observes the exact offers (or quality).

4.3 Asymmetric Focus On Different Projects

Unlike the rest of this chapter, this section considers the case when the decision maker can focus his attention on different projects; under the motivation of coarse communication this is represented by reallocating an agent, so that a single project is inspected more than once using different thresholds. This means that the possible realisations of utility for one project are partitioned more finely while for another, they are not partitioned at all.

Conjecture 4.1 *Asymmetric optimal partitioning consists of internal intervals which are not contained within an interval on another project. This implies that intervals must be spread as evenly as possible over the projects.*

Proof. Appendix A.5 sketches a proof that optimally no internal interval should lie strictly within the limits of an interval on another project. Appendix A.6 sketches a proof that optimally, no internal interval should lie weakly within the limits of an interval on another project. This is done using general arguments to provide intuition that extends to the case when intervals are spread over more than two projects. Therefore, where possible, intervals should overlap. ■

Conjecture 4.1 implies that the optimal asymmetric partitioning when $k+N$ intervals are spread over N projects must have the optimal structure illustrated in Figure 4.3.

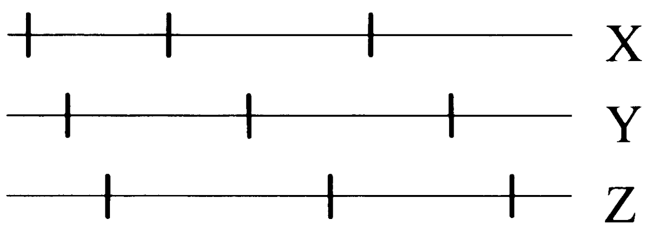


Figure 4.3: Optimal interval structure

As illustrated in Figure 4.3 the requirement that internal intervals overlap means that intervals must be constructed in such a way that they are spread as evenly as possible over different projects. Therefore the solution in Proposition 4.2 when there are N and $2N$ intervals is more generally optimal, as the expected error would increase if the decision maker left one project unpartitioned and partitioned another twice.

Conjecture 4.1 appears to contrasts with Fryer and Jackson (2004). In their model, a majority group is partitioned more finely and a minority group more coarsely. This result can be attributed to a difference in the cost structure of introducing additional intervals, as in their paper, individuals think through a limited number of categories. Considering the case of symmetric communication of Proposition 4.1, an additional category can be applied to any number of projects. In contrast, the motivation underlying Conjecture 4.1 is based on the precision of information acquisition in a screening process, represented by the *total* number of partitions; applying a single additional category to N projects requires N additional partitions. Within the framework of this thesis, their result that a majority group should be partitioned more finely is natural, as adding a category to the larger group means there are more additional partitions and therefore more information is communicated.

Chapter 5

Partially Asymmetric Communication with N Options

Chapters Three and Four investigated the optimal screening (or communication) process in two extreme cases of fully symmetric and fully asymmetric communication. The latter case is complex and requires the decision maker to identify every agent uniquely and remember every different partition used. This chapter considers the partially asymmetric case, between these extremes, where the decision maker can identify which group an agent (or project) is in. Continuing the analogy with the housing-selling problem developed in Chapter Four, this is the simultaneous equivalent of a heuristic proposed by Lee, O'Connor and Welsh (2006) to describe behaviour in an experimental secretary problem. They point out that "*These sorts of heuristics seem likely to have complexity that lies somewhere between that of the biased optimal and (single) threshold heuristics. It may well be the case that human performance is best explained by an account that is more sophisticated than the (single) threshold heuristic, but does not have the full complexity of the biased optimal approach.*"[1]

One motivation for such a grouping is that the decision maker identifies each project by an economically irrelevant characteristic, and uses different thresholds for projects that are associated with different groups. The division into groups of different sizes is exogenous, and Conjecture 5.1 provides intuition relevant to this case. In the optimal partially asymmetric screen, projects in the majority group face a higher threshold but are chosen in preference to projects in the minority group in the decision stage. Conjecture 5.2 considers the case when groups are created endogenously. For example, the decision maker could assign projects to be evaluated by relatively pessimistic

[1] Quoted from Lee, M. D., O'Connor, T. A., & Welsh, M. B (2006) "Human Decision-Making on the Full Information Secretary Problem" page 6.

accountants or optimistic marketing managers.[2] Alternatively, there could be a range of project characteristics on which to base a grouping, but for simplicity it is assumed that only two such groups are formed. Even though the decision maker could make the two groups equal in size, such a grouping is suboptimal; it is globally optimal to form majority and minority groups endogenously, and projects in the minority group have a lower probability of being chosen. These insights will be related to some models from the economic theory of discrimination, where the use of biased screening tests is typical. Not only does this analysis suggest that a biased screen might be optimal, but it also implies that this could lead to discrimination against a *minority*. For example, the intuitions developed in this chapter extend to the model of Cornell and Welch (1996), meaning that discrimination is optimal *even without* their assumption that the decision maker gets less accurate signals from the minority group.[3] The analysis can also explain both discrimination and the responsiveness of callbacks to quality in field experiments in which fictitious résumés are submitted to companies,[4] without requiring that there be any exogenous difference between individuals other than that they are identified with either a minority or majority group.

5.1 Optimal Partially Asymmetric Communication

As in Chapters Three and Four the possible project utility values are identically, independently and uniformly distributed. Figure 5.1 shows how the projects (labelled X_i) are divided into two groups, B (containing n projects) and A (containing $m = N - n$ projects). Projects in group B face a stricter screening threshold relative to those in group A, so threshold $b > a$. In the decision stage, *good* projects in group B have a higher expected value than *good* projects in group A. The problem is to maximise the expected value by choosing a, b and n optimally, where n is the number of projects in group B. When the groups are given exogenously, n is chosen from just two alternatives (i.e. to make B or A the majority group).

[2] It is assumed the decision maker finds or trains suitably optimistic or pessimistic agents.

[3] This literature will be discussed at the end of this chapter.

[4] Bertrand and Mullainathan (2004) find evidence of discrimination by submitting the same fictitious résumés with either African American or White sounding names. This will be discussed in more detail at the end of the chapter.

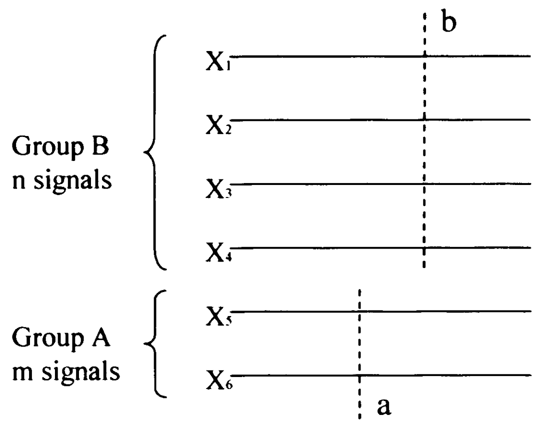


Figure 5.1: Optimal partially asymmetric intervals when N=6

In the following example there are N projects split into two groups B and A.

Group B: $\mathbf{X}_b = \{X_1 , \dots , X_n\}$ of n projects

Group A: $\mathbf{X}_a = \{X_{n+1} , \dots , X_N\}$ of $m = N - n$ projects

The event that all projects in group B have *low* realisations is defined as X_B^L; if some project in B has a *high* realisation this is defined as the event X_B^H. The event that all projects in group A have *low* realisations is defined as X_A^L; if some project in A has a *high* realisation this is defined as the event X_A^H. The probabilities and expected utility can be calculated conditional on the signal profiles resulting from these events and are shown in Table 5.1.

s	Choice	$MaxE[V \mid s]$	$\Pr[s]$
X_B^H , $X_A^{\cdot}$	X_B^H	$\frac{1+b}{2}$	$1 - b^n$
X_B^L , X_A^H	X_A^H	$\frac{1+a}{2}$	$b^n\left[1 - a^m\right]$
X_B^L , X_A^L	X_B^L	$\frac{b}{2}$	$b^n a^m$

Table 5.1: Derivation of expected value

Proposition 5.1 *When the decision maker chooses between* $(n+m)$ *options divided into* 2 *groups,* B *and* A*, optimal partially asymmetric communication consists of dividing possible realisations of utility for projects in group* B *into intervals* $\{X_B^L , X_B^H\} = \{[0,b) , [b,1]\}$ *and for projects in group* A*,* $\{X_A^L , X_A^H\} = \{[0,a) , [a,1]\}$*.* a *and* b *satisfy Equations* 5.1 *and* 5.2*.*

Proof. Substituting the probabilities and conditional expectations in Table 5.1 into $E[V] = \sum_S E[V \mid s]\Pr[s]$ gives $E[V] = \frac{1}{2}\left[1 + b + (a-b)\,b^n + [b - 1 - a]\,b^n a^m\right]$.

Therefore:

$$\frac{dE[V]}{da} = \frac{b^n}{2}\left[1 - a^m - (a-b+1)\,ma^{m-1}\right] = 0 \tag{5.1}$$
$$\frac{dE[V]}{db} = \frac{b^{n-1}}{2}\left[\frac{1}{b^{n-1}} - b + (a-b)\,n + ba^m - (a-b+1)\,na^m\right] = 0 \tag{5.2}$$

As $n \to \infty$ and $m \to \infty$, $a \to 1$ and $b \to 1$. An exact analytical solution is not possible because solving Equation 5.2 requires finding the root of a polynomial of degree five or greater, for which no general solution exists. The problem is further complicated by the requirement that m and n be integers. ■

An approximation for a and b when n and m are large can be derived from Equations 5.1 and 5.2:

$$a \approx [m+2]^{-\frac{1}{m}} \tag{5.3}$$
$$b \approx [n(1-a)+1]^{-\frac{1}{n}} \approx \left[n+1-n\,[m+2]^{-\frac{1}{m}}\right]^{-\frac{1}{n}} \tag{5.4}$$

The conjectures in the remainder of this chapter can be demonstrated using such approximations. However, the approach taken is to use numerical methods to calculate the exact optimal values of a and b and motivate the ideas using examples. This is to demonstrate that it is the model, rather than the approximation, which leads to Conjectures 5.1 and 5.2, as expected errors will often be small. **The ideas in this chapter will be analysed numerically using the example of $N = 6$, then illustrated for $N = 50$. Finally Figures 5.5 and 5.7 will illustrate how the properties of the optimal solutions depend on N.**

Equations 5.1 and 5.2 require that $0 < n < N$. If $N = n$ or $n = 0$ the optimisation problem reduces to the symmetric case,[5] giving a solution of either $b = \left(\frac{1}{N}\right)^{\frac{1}{N-1}}$ or $a = \left(\frac{1}{N}\right)^{\frac{1}{N-1}}$ respectively. When $n = m = 1$ Equation 5.1 gives the first order condition $b = 2a$ and Equation 5.2 gives the condition $2b = a + 1$, so Proposition 5.1 generalises Proposition 3.4.

Example 5.1 *Projects can be identified as being in one of two exogenous groups B and A of sizes $n = 4$ and $m = 2$ respectively. Optimal partially asymmetric communication consists of dividing possible realisations of utility for projects in groups B and A into intervals* $\{X_B^L\,,\ X_B^H\} = \{[0, 0.7545)\ ,\ [0.7545, 1]\}$ *and* $\{X_A^L\,,\ X_A^H\} =$

[5] When $n = 0$ the problem needs to be respecified slightly as the payoff if all signals are low will be $\frac{b}{2}$ rather than $\frac{a}{2}$.

$\{[0, 0.5013)\ ,\ [0.5013, 1]\}$ *respectively. This solution, which is shown in Table* 5.2, *generates an expected error of* 0.0513.

b	a	n	m	V	ϵ
0.7545	0.5013	4	2	0.8058	0.0513

Table 5.2: Optimal partially asymmetric communication when n=4 and m=2

It is straightforward to verify that the values of a, b, n and m satisfy Equations 5.1 *and* 5.2 *in Proposition* 5.1.

Figure 5.2 illustrates the solution in Table 5.2. The partially asymmetric optimal values of $a = 0.5013$ and $b = 0.7545$ are illustrated as dashed lines. The fully asymmetric optimal interval boundaries, derived in Example 4.2, are denoted by squares.

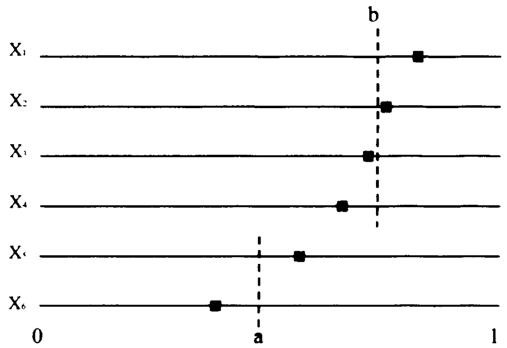


Figure 5.2: Comparison of asymmetric and partially asymmetric communication

Although in the optimal fully asymmetric case the labelling of projects is arbitrary, in Figure 5.2 the optimal partitions are ordered in descending strictness of the thresholds, a_i, for comparison with the partially asymmetric case. The fully asymmetric optimal partitions are more spread out in group A than group B (Figure 5.4 illustrates this point more clearly when N is larger).

Example 5.2 *Suppose that in Example* 5.1 *the decision maker chose the smaller group to face a stricter threshold, so the two groups B and A are of sizes $n = 2$ and $m = 4$. Optimal partially asymmetric communication consists of dividing possible realisations of utility for projects in group B into intervals* $\{X_B^L\ ,\ X_B^H\} = \{[0, 0.7940)\ ,\ [0.7940, 1]\}$ *and for projects in group A,* $\{X_A^L\ ,\ X_A^H\} = \{[0, 0.6311)\ ,\ [0.6311, 1]\}$. *This solution, which is reported in Table* 5.3, *generates an expected error of* 0.0534.

b	a	n	m	V	ϵ
0.7940	0.6311	2	4	0.8038	0.0534

Table 5.3: Optimal partially asymmetric communication when n=2 and m=4

It is straightforward to verify that the values of a, b, n and m satisfy Equations 5.1 *and* 5.2 *in Proposition* 5.1.

A comparison of Examples 5.1 and 5.2 leads to two important observations. Firstly, even when a and b are adjusted to their optimal values, it is better to subject the larger group to the stricter threshold. In Example 5.1 the expected error is 4.1% lower than in Example 5.2. Secondly, b and a are *both* significantly larger in Example 5.2.

Conjecture 5.1 *When partitioning is based on projects being identified in one of two exogenous groups, it is better to partition the larger group using the higher threshold, b, and the smaller group using the lower threshold, a, providing $n + m \geq 4$.*[6] *If a project in group B has a high realisation then it is chosen in preference to projects in group A in the decision stage.*

Although Proposition 5.1 cannot be solved explicitly it is possible to illustrate that Conjecture 5.1 holds in approximation. However, as the errors in these problems are small, this thesis uses numerical methods to solve Equations 5.1 and 5.2 exactly. This section will proceed by introducing Example 5.3 and Conjecture 5.2, which is closely related to Conjecture 5.1. Both will be illustrated diagrammatically for the case where $N = 50$ and the common motivation for the conjectures will be discussed.

Example 5.3 *Suppose that in Example* 5.1 *the decision maker was able to choose the grouping of the six projects endogenously, possibly based on different characteristics.*[7] *Given $N = 6$, the size of groups can be denoted by n and m ($m = N - n$), where the group of n projects faces the higher threshold b. The minimised expected error under optimal partially asymmetric communication when $n = 4$ was calculated in Example* 5.1 *as $E[\epsilon] = 0.0513$. When $n = 2$ Example* 5.2 *showed that minimised $E[\epsilon] = 0.0533$. When $n = 0$ or $n = 6$ the optimal symmetric partition in Example* 4.1 *gives $E[\epsilon] = 0.0660$. Table* 5.4 *summarises these results and completes for the cases when $n = 1$, $n = 3$ and $n = 5$.*

n	0	1	2	3	4	5	6
Min $E[\epsilon]$	0.0660	0.0556	0.0534	0.0516	0.0513	0.0540	0.0660

Table 5.4: Optimal partially asymmetric communication when n is endogenous for N=6

For each value of n the optimal values of a and b (which are set optimally given n) are omitted to conserve space.

Example 5.3 can be used to illustrate Conjecture 5.1, that it is optimal to partition the larger group using stricter thresholds, as Table 5.4 shows that when a and b are set optimally, the minimised expected error is less when $n = 5$ (and therefore $m = 1$)

[6] When the number of projects is very small, meaning $n + m < 4$, the chance of choosing a project in either group is relatively high and Conjectures 5.1 and 5.2 may not hold.

[7] For simplicity it is assumed that only two groups are formed.

than when $n = 1$ (and $m = 5$), and less when $n = 4$ than when $n = 2$.[8] However, Example 5.3 also illustrates another interesting point. Setting $n = 4$ and $m = 2$, leading to optimal partially asymmetric communication developed in Example 5.1, is actually optimal when n is endogenous. Therefore the expected error is less when there is a minority and a majority group than when both groups are of the same size ($n = m = 3$). So if groups are partitioned based on a single characteristic, it is optimal to choose a characteristic creating groups of different sizes. As with Conjecture 5.1, this holds generally providing $n + m > 4$.

Conjecture 5.2 *When partitioning is based on N projects being identified in one of two endogenously formed groups of sizes n and m, when $N \geq 4$ it is better to use a characteristic so that group B is larger than group A (where threshold $b > a$). If a signal in group B is* high *then it is chosen in preference to projects in group A in the decision stage.*

Although Proposition 5.1 cannot be solved explicitly it is possible to illustrate that Conjecture 5.2 holds in approximation. However, as the error values in these problems are typically small, this thesis uses numerical methods to solve Equations 5.1 and 5.2 exactly. Example 5.4 is equivalent to Example 5.3 for the case when $N = 50$. The results are clearer graphically when N is larger.

Example 5.4 *Given $N = 50$ the size of groups A and B can be denoted by n and $m = N - n$ respectively, where group B faces the higher threshold, b. Whether n is defined exogenously as in Conjecture* 5.1 *or endogenously as in Conjecture* 5.2, *expected error can be calculated conditional on n. For each value of n, Equations* 5.1 *and* 5.2 *can be used to calculate the optimal values of a and b which in turn are used to calculate the minimum expected error. Note that when $n = 0$ or $n = 50$ Proposition* 4.1 *shows that under optimal symmetric communication, expected error equals* 0.0278. *This provides an upper bound for the expected error once partial asymmetry is introduced.*[9] *The minimum expected errors are illustrated in Figure* 5.3.

[8] As demonstrated in Example 5.2.

[9] Partially asymmetric communication (or screening) could always achieve this bound by setting $n = m$.

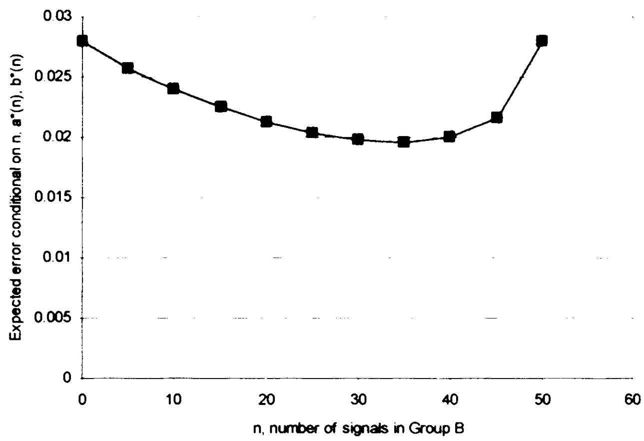


Figure 5.3: Minimum expected error conditional on n when N=50

The graph in Figure 5.3 shows the lower bound on expected error that occurs when a and b are set optimally. Conjectures 5.1 and 5.2 are represented by the skew in the graph. The skew means that for any given division of N into groups N_1 and N_2, it is optimal to set $n = \max[N_1, N_2]$. For example, Figure 5.3 shows that when $N_1 = 10$, setting $n = 10$ gives an expected error of 0.0240 while setting $n = 40$ gives a lower expected error of 0.0200. The skew implies that this is true for any $n < 25$, illustrating Conjecture 5.1. Secondly, because of the skew, the global minimum expected error occurs when $n = 34$ and $m = 16$ (rather than at $n = \frac{N}{2} = 25$). This solution is shown in Table 5.4 and illustrated in Figure 5.4, and demonstrates Conjecture 5.2.

b	a	n	m	V	ϵ
0.9456	0.8346	34	16	0.9608	0.0196

Table 5.5: Optimal partially asymmetric communication when n=34 and m=16

As in Figure 5.4, the optimal fully asymmetric partitions (shown as squares) are ordered in descending strictness of thresholds, a_i, for comparison with the partially asymmetric case illustrated by the dashed lines at $b = 0.9456$ and $a = 0.8346$.

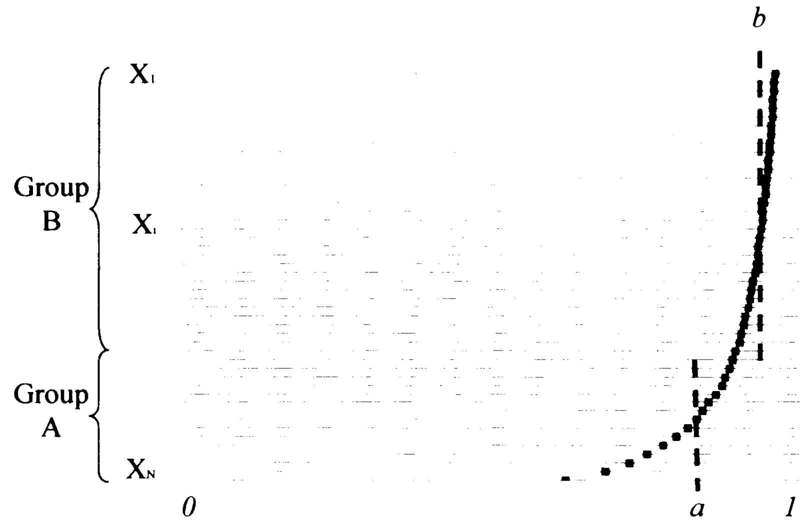


Figure 5.4: Optimal partially asymmetric communication when N=50

Figure 5.4 shows firstly that the optimal group sizes are very different: group B contains more than twice as many projects as group A ($n = 34$ and $m = 16$). Secondly, the fully asymmetric optimal partitions are more spread out in group A, so in a non-technical sense a larger group B and smaller group A is closer to the fully asymmetric optimal solution than if a minority faced the stricter screening threshold in group B.

The motivation underlying the optimal unequal division into groups is that when m and n are large the chance of some project in group B being chosen is approximately $1 - b^n(1 - a^m)$ while the chance of some project in group A being chosen is $b^n[1 - a^m]$, because as well as the difference between a and b, projects with a *high* realisation in group B are chosen in preference to projects in group A. This is rational as in the decision stage the project with a realisation that exceeds the higher threshold has a greater expected value. In the optimum given in Table 5.5 and illustrated in Figure 5.4, even if the groups were rebalanced so that $n = m = 25$, with $b = 0.9456$ and $a = 0.8346$, the probability of choosing some project in group B is 0.7557 while the probability of choosing a project in group A is only 0.2443.[10]

Starting from the case in which the decision maker has N options divided into two groups B (containing n projects) and A (containing m projects), a useful thought experiment is to consider whether to allocate an additional project, labelled X_g, into group B or group A. Addition of X_g to group B increases the expected value to $\frac{1+b}{2}$

[10] If a and b were chosen optimally for the case where $n = m = 25$ the probabilities of choosing some project in group B or A would be 0.7440 and 0.2556 respectively.

in the case that X_g is high while all other signals in group B are *low*. This expected gain can be expressed:[11]

$$\begin{aligned} E\left[G_B\right] &= \Pr\left[X_{gH}\right]\Pr\left[X_B^L\right]\left[\frac{1+b}{2} - E\left[V \mid X_B^L\right]\right] \\ &= (1-b)\,b^n\left[\frac{1+b}{2} - \left(\frac{1+a}{2}(1-a^m) + \frac{b}{2}a^m\right)\right] \end{aligned}$$

On the other hand, addition of X_g to group A increases the expected value to $\frac{1+a}{2}$ rather than $\frac{b}{2}$ when X_g is *high*, but only if all signals in both groups are *low*. The expected gain can be expressed:

$$\begin{aligned} E\left[G_A\right] &= \Pr\left[X_{gH}\right]\Pr\left[X_B^L\right]\Pr\left[X_A^L\right]\left[\frac{1+a}{2} - E\left[V \mid X_B^L, X_A^L\right]\right] \\ &= (1-a)\,b^n a^m\left[\frac{1+a}{2} - \frac{b}{2}\right] \end{aligned}$$

When the groups are of similar sizes (and $n+m \geq 4$) and a and b are set optimally,[12] the dominant effect is that an additional project in group B adds value when all other signals in group B are *low*, but allocating the project to group A only increases expected value if all signals in *both* groups A and B are *low*. To some extent this is offset as the expected gain from allocating a project to group A is greater conditional on it being chosen, and the chance of a project in group A having a *high* realisation is greater than if it were allocated to group B. When n and m are set optimally, these exactly offset the dominant effect. However, when $n \approx m$, because a project which is rarely chosen adds little value, there is a correlation between the probability of choosing a project (which is higher in Group B) and the amount it adds to expected value. This is demonstrated in Example 5.5.

Example 5.5 *Consider the case when* $N = 50$ *and* $n = m = 25$. *Optimally* $b = 0.9484$ *and* $a = 0.8759$ *giving* $E[V] = 0.9601$ *and* $E[\varepsilon] = 0.0203$. *At these values,* $E[G_B] = 0.00073$ *while* $E[G_A] = 0.00056$. *This imbalance represents the fact that* $n = m = 25$ *is not a local optimum when* n *is endogenous. Increasing* n *so that* $n = 26$ *and* $m = 24$ *increases the expected value to* $E[V] = 0.9602$.

Although this chapter focuses on finding the global optimum, as n is an integer there are often multiple local optima. For example, when $N = 6$, $n = m = 3$ is a local

[11] It is possible to show that setting $E[G_B] = E[G_A]$ is equivalent to maximising $E[V]$ in Proposition 5.1 with respect to n, which differentiates to give $\frac{dE[V]}{dn} = \frac{b^n}{2}\left[(a-b)\ln b + [b-1-a]\,a^m \ln\left(\frac{b}{a}\right)\right]$.

[12] This is required for the result that more projects should be allocated to the group facing a higher threshold. For example, if $b = 1$ or $b = 0$ then clearly all signals should be allocated to group A, as screening in group B is totally uninformative. Alternatively, if $a = b$ then division into groups becomes irrelevant and it can be verified that $E[G_B] = E[G_A] = \frac{1-b}{2}b^N$, which is independent of n and m.

optimum because given the optimal $a = 0.5780$ and $b = 0.7728$ it is not optimal to change n and m (even though $n = 4$ and $m = 2$ gives the unique global optimum). When $N = 50$, the changes in a and b resulting from a unit change in n and m is smaller and so $n = m = 25$ is not a local optimum as shown in Example 5.5.

When $N = 6$, Example 5.3 showed that optimal endogenous choice of n was $n = 4$ (and therefore $m = 2$) which minimises expected error when a and b are set optimally, so the optimal proportion of projects in group B was $\frac{n}{N} = \frac{4}{6} = 0.3333$. When $N = 50$ the optimal solution was the minimum illustrated in Figure 5.4, where $n = 34$ and $m = 16$, so the optimal proportion of projects in group B was 0.68. Figure 5.5 shows how this ratio varies depending on N. As N increases it is optimal to allocate an increasing proportion of projects to group B (recalling that threshold $b > a$).

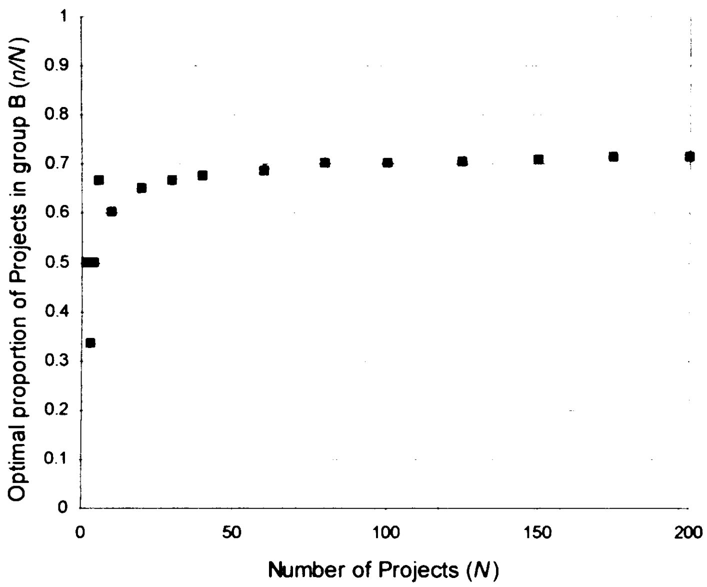


Figure 5.5: Optimal proportion of projects in group B

Figure 5.5 supports the intuition underlying Conjecture 5.2. When N and hence m is small, an additional project has a reasonable chance of being chosen in either group when $n \approx m$, but as N and therefore m increases, the probability that all projects in group A have *low* realisations becomes sufficiently small that the project is better allocated to group B.

5.2 Analysis of Optimal Partially Asymmetric Screening

This section begins with the conjecture that whether groups are formed exogenously (as in Conjecture 5.1) or endogenously (as in Conjecture 5.2), projects in the majority group have a higher probability of being chosen *a priori* when a and b are set optimally. Using the screening motivation, this is because although projects in the majority group are subjected to a tougher test, passing means that they are more likely to be chosen in the decision stage. Projects in the minority group face an easier test, but even if they pass, they may be rejected in favour of a member of the majority group which has passed a tougher test and therefore has a higher expected realisation. This is demonstrated in Example 5.6 for the case when $N = 6$ and Figure 5.7 will illustrate that this result holds for all values of $N \geq 4$ when n is endogenous.

Proposition 5.2 *Consider when N projects are divided into two groups B (containing n projects) and A (containing m projects) where projects in each group are partitioned using thresholds b and a (where $b \geq a$) respectively. The a priori probability of a specific project in either group being chosen is illustrated in Table* 5.6. *It is assumed that when indifferent at the decision stage, the decision maker randomises between projects.*[13]

	Unbiased Case	$X_i \in X_B$ *(Group B)*	$X_i \in X_A$ *(Group A)*
$\Pr(X_i \text{ chosen} \mid X_i \text{ high})$	$\frac{1}{N(1-\bar{a})}\left[1-\bar{a}^N\right]$	$\frac{1}{n(1-b)}\left[1-b^n\right]$	$\frac{1}{m(1-a)}b^n\left[1-a^m\right]$
$\Pr(X_i \text{ chosen} \mid X_i \text{ low})$	$\frac{1}{N}(\bar{a})^{N-1}$	$\frac{1}{n}b^{n-1}a^m$	0
$\Pr(X_i \text{ chosen})$	$\frac{1}{N}$	$\frac{1}{n}\left[1-b^n\left(1-a^m\right)\right]$	$\frac{1}{m}b^n\left[1-a^m\right]$

Table 5.6: *Probability a project is chosen conditional on group and realisation*

Conjecture 5.3 *Under optimal partially asymmetric communication, where a and b are set optimally and $n \geq m$, projects in group B, which face the higher threshold b, have a greater probability of being chosen a priori than projects in group A.*

Conjecture 5.3 is illustrated in Example 5.6 for the case when $N = 6$, using the probabilities derived in Proposition 5.2.

Example 5.6 *Consider the case when $N = 6$. The probabilities of a project being chosen conditional on it being in the majority group (B) or the minority group (A) are listed in Table* 5.7 *and illustrated in Figure* 5.6. *They are calculated using the optimal partially asymmetric solution when $N = 6$ given in Table* 5.2. *It is assumed that when the decision maker is indifferent between projects at the decision stage he*

[13] The decision maker may randomise between two projects in group B if both signal *high* realisations, but is never indifferent between a project in group A and one in group B.

randomises between them.

	Unbiased Case	$X_i \in X_B$ *(Group B)*	$X_i \in X_A$ *(Group A)*
$\Pr(X_i \text{ chosen} \mid X_i \text{ high})$	0.4889	0.6883	0.2433
$\Pr(X_i \text{ chosen} \mid X_i \text{ low})$	0.0278	0.0270	0
$\Pr(X_i \text{ chosen})$	$\frac{1}{6}$	0.1893	0.1213

Table 5.7: Probability of a project being chosen when N=6

The probability of a player in either group A or group B being chosen can be analysed relative to the benchmark of symmetric communication. This is a comparison of optimal partially asymmetric communication in Table 5.2, where $a = 0.5013$ and $b = 0.7545$, and optimal symmetric communication in Example 4.1, where $\bar{a} = 0.6988$.

Figure 5.6 illustrates the probability of a project in group B (upper diagram) or group A (lower diagram) being chosen conditional on its realisation, shown on the x axes. The vertical dashed line illustrates the optimal symmetric partition at $\bar{a} = 0.6988$, and the horizontal dashed lines show the probability of a project being chosen conditional on its realisation in the symmetric case. The solid black lines illustrate the probability a project is chosen conditional on its realisation in the optimal partially asymmetric case. The probabilities illustrated are given in Table 5.7.

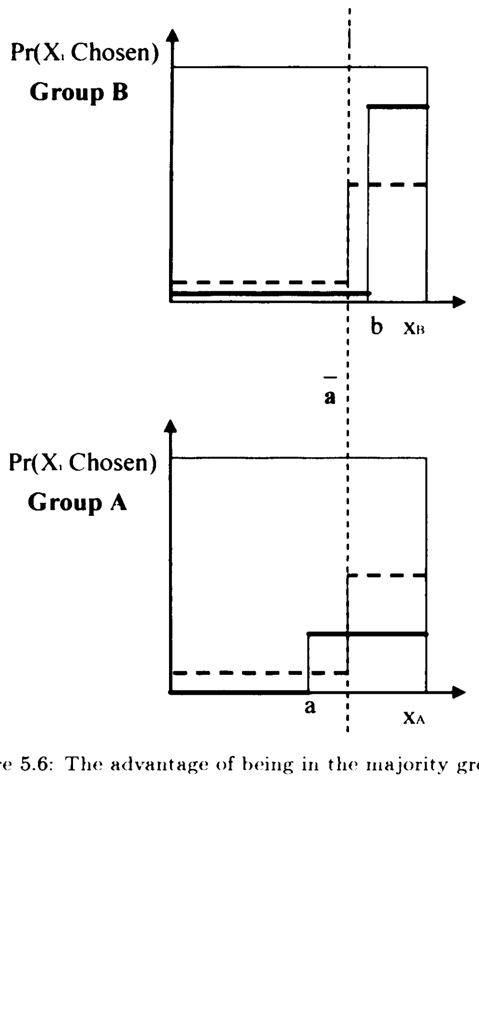


Figure 5.6: The advantage of being in the majority group

As discussed at the start of this section, in the optimal partially asymmetric case a project in the minority group A faces a less strict screening threshold, but is only chosen in the decision stage if all signals in group B are *low*. Therefore compared to the symmetric case, projects with *high* realisations in group B are more likely to be chosen, while projects with *high* realisations in group A are less likely to be chosen. However, not all realisations in group B are more likely to be chosen under asymmetry; in the range $[\overline{a}, b)$, as the threshold b becomes higher, realisations will move from a *high* interval to a *low* interval. Conversely, in the range $[a, \overline{a})$ projects in group A are more likely to be chosen than under symmetric screening, as the threshold a has fallen. However, the *a priori* chance of a specific project being chosen is higher for projects in group B as the increased chance of being chosen in the decision stage outweighs the lower probability that the realisation is *high*. In Example 5.6 the *a priori* probability of a project being chosen is 0.1893 if it is in group B but only 0.1213 if it is in the smaller group A, compared to a common probability of 0.1667 under symmetric screening.

In addition, for projects in group B, moving from a *low* to a *high* realisation increases the probability of being chosen by 0.6613, while moving from a *low* to a *high* realisation for projects in group A only increases the chance that a specific project is chosen by 0.2433. In a model with error (which smooths the probabilities) the returns to having a higher realisation are greater for projects in the majority group. This result is important and will be discussed in relation to résumé studies and the economics of discrimination at the end of this chapter.

Figure 5.7 shows the probability of a project being chosen conditional of whether it is in group A or B in the partially asymmetric case compared to symmetric communication for values of N other than $N = 6$. It is assumed in this figure that n, a and b are set optimally given N.

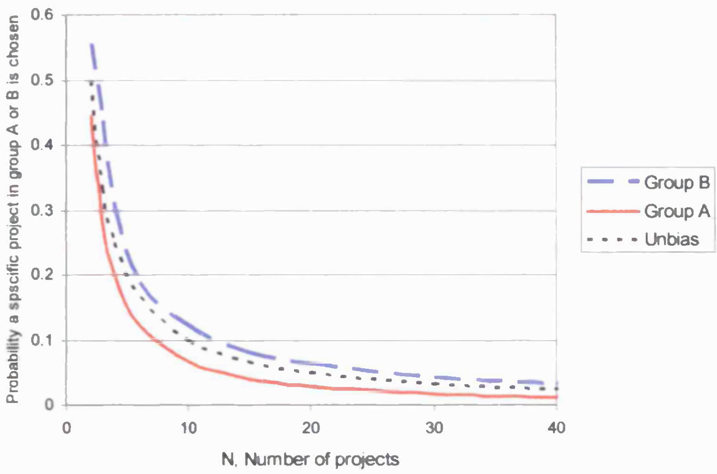


Figure 5.7: Probability of being chosen conditional on group when n is endogenous

Figure 5.7 illustrates that the *a priori* chance of a specific project being chosen is always higher for projects in group B, the group facing the stricter screening threshold, therefore Conjecture 5.3 holds generally when n is chosen endogenously. In addition, as N becomes larger the difference in the probability of being selected (between groups A and B) remains relatively significant, although the probability of a specific project in group B being chosen becomes closer to the symmetric case. This is because an increasing proportion of projects are allocated to group B, as illustrated in Figure 5.5.

5.3 Discussion

The contribution of this chapter is to show that within a class of screening processes, it is optimal to introduce asymmetry in a manner that reduces the *a priori* probability that a project in a minority group is chosen. In the economics of discrimination it is typical for a single individual to perform the screen; this model could be interpreted as an optimal boundedly rational screening process which motivates the automaticity of discrimination when there are information acquisition constraints (a closely related automatic discriminatory process based on optimal category formation is developed in Fryer and Jackson, 2004). This is optimal despite there being no economic difference between the groups. Although the characteristic used to group individuals is arbitrary, race is perhaps focal as an identification measure and becomes available early in the search process when a screening procedure (rather than pairwise comparisons) and coarse information are best motivated. This links the analysis in this chapter to recent studies of *résumé screening*, in which fictitious résumés are submitted to firms, while the names of candidates, which indicate their cultural background, are varied.

Discrimination is defined as *allowing a member of a minority group to be treated differently from a member of the majority despite them having identical characteristics*. There are two main competitive approaches to the economics of discrimination.[14] The first is *taste-based discrimination*, which has limited relevance to this thesis. Formalised by Becker (1957), this allows *discrimination coefficients* to enter the utility functions of firms, other employees or customers. This faces two main criticisms: firstly it assumes discrimination rather than explaining it, and once it is assumed that firms do not maximise profits it is necessary to question why they do not include factors such as growth or effort in their optimisation problems. Secondly, discrimination in these models is costly, so discriminating firms would not survive under perfect competition.[15]

A second approach to explaining discrimination is *statistical discrimination*. This may occur when an employer has imperfect information about the productivity of a worker, and uses ethnic stereotypes to minimise the information costs involved in hiring decisions.[16] For example, if there is a correlation between racial characteristics and unobserved elements of productivity, then an optimising firm would use the racial characteristics in the hiring process (Arrow, 1973). Unlike taste-based discrimination, in this case it is the firms which do not discriminate that lose out. Statistical discrimination was originally motivated in racially biased hiring by Phelps (1972) in a model which was developed by Aigner and Cain (1977), who argue that if a minority group has a noisier signal of productivity (in testing), then the productivity of an individual worker would be judged more on group characteristics, lowering the return to investment in human capital, including unobservable *work skills*. Lundberg and Startz (1983) incorporate human capital investment explicitly in a model where employers are able to assess the productivity of a minority group less accurately. They show that a separating equilibrium can arise in which discrimination can persist even in a competitive situation.

Therefore as well as the case when productivity differs across groups, statistical discrimination can occur if signals of productivity are less informative for a minority group, due to less accurate testing, or if members of the minority group underperform in testing. In contrast, the model presented in this thesis shows that discrimination could arise even when there are no exogenous differences between members of the

[14] These are also collective explanations where cartels form. Firstly, these could be constructed so that a minority gains, and secondly, it is hard to imagine millions of people with different incentives and lifestyles forming a single tacit cartel.

[15] If firms have market power they could discriminate indefinitely but empirically there is little evidence that monopolies and oligopolies discriminate more. Besides, minorities could simply work in competitive industries.

[16] It is assumed that there is some sunk cost involved in hiring, so a costly search process is worthwhile.

minority or majority group, other than the size of the group with which they are identified. At this point it is important to make a distinction between *discrimination*, which can be supported as a separating equilibrium of a search or signalling problem with multiple equilibria, and *minority discrimination*, where discrimination is against a minority group specifically. While a model of coordination failure is theoretically appealing, as it does not require exogenous assumptions on how groups differ, unlike the model presented in this thesis, it cannot explain why it is not the *majority* who is discriminated against.

The analysis presented in this thesis applies directly to a model developed by Cornell and Welch (1996), who also focus on discrimination in hiring rather than wages. In their model each worker has an identically, independently distributed *quality*. Firms observe binary signals which are correlated with the *quality* of a worker. Specifically *quality*, Q, is uniformly distributed over $[0, 1]$ and the probability of a *high* signal is Q and a *low* signal $1 - Q$. They assume that an interviewer gets additional signals if the worker is from the same cultural background. There is a higher variance in the conditional quality distribution of workers from the same background,[17] making it more likely that a worker from the same background is chosen. Although there are no *a priori* productivity differences between groups, there are differences in the information structure. In this situation, bias screening[18] would be optimal and would favour a majority, so discrimination would be optimal even without Cornell and Welch's assumption that the information structure differs between groups.

This approach can be contrasted to Fryer and Jackson (2004) who argue that bias arises when individuals form categories, which minimise the sum of within group variation in a broader context than the economic decision at hand. This means that less frequent observations tend to be grouped more coarsely and so the categories containing them are more heterogeneous.[19] They give an example in which workers are divided on two dimensions: each is either *Good* or *Bad* and *Black* or *White*. When grouped using three categories, if the number of *Black* workers is relatively small, the categories which minimise the sum of within category variation are *Good White*, *Bad White* and *Black*. In an economic decision, categorising groups in this way leads to *Good White* workers being preferred to *Black* workers as pooling *Good Black* workers with *Bad Black* workers creates an average expected marginal productivity.

[17] Fewer signals for the minority group mean that expected quality deviates less from the prior.

[18] Bias could be introduced in the manner of Calvert (1985), which was discussed in Chapter Three.

[19] This contrasts with Cornell and Welch (1996) if society is evenly mixed, as members of a minority group would also partition the majority group more finely. If there is a degree of segregation, minority individuals in Fryer and Jackson (2004) might group those from their own background more finely.

The outcome and implications of the model in this thesis and Fryer and Jackson's are very similar, as in both cases good minority workers face less strict thresholds than the majority group.[20] Therefore average workers are *categorised* with excellent workers amongst the minority, reducing the expected utility from choosing an individual with a *good* signal, so an individual from the majority group with a *good* signal would be chosen in preference.

Despite these similarities, the model presented in this thesis differs from those presented in Cornell and Welch (1996) and Fryer and Jackson (2004) in two important aspects. Firstly, *minority discrimination* is explained entirely within the economic sphere, without differences arising due to difficulties with cross-cultural communication or it being desirable to categorise non-economic characteristics differently. Secondly in these papers, screening of minority groups is assumed to be coarser (or less accurate). In contrast, Conjecture 4.1 argued that for a given number of partitions, it is optimal to spread them as evenly as possible across projects. This can be interpreted as giving the same attention to each applicant (of both minority and majority groups). The type of bias that is optimal in the screening process described in this thesis is to bias the cutoff productivity levels directly rather than biasing the *accuracy* of the tests between groups. In résumé screening however, it is hard to imagine why résumés would be grouped less coarsely or be less accurate for a minority group.

Although this is not intended as an applied thesis, it is closely related to some recent interesting empirical results on résumé screening, which are particularly relevant as both coarse signals and screening are best motivated early in the search process. Bertrand and Mullainathan (2004) study racial discrimination in US labour markets by submitting fictitious résumés to companies. Résumés are randomly assigned an African American or White sounding name, and they find that White sounding names receive 50% more callbacks for interview, a result that persists across occupation, industry and employer size. In addition, they found that callbacks are also more responsive to résumé quality for White names than African American names, and the gap is largest for those with high skill or education levels. Both of these facts are consistent with the model presented in this chapter: the probability of being hired increases more for high quality members in the majority group than in the minority group, and the difference between the probability of being selected is greater between high quality workers in the minority and majority groups than it is for low quality workers.

[20] The reasoning behind this outcome is different however: in the latter case, this is because they do not face a threshold at all.

Chapter 6

Conclusion

This thesis investigated how a decision maker might optimally screen projects to maximise expected utility. In Chapter Three the screening heuristic was introduced as a coarse communication mechanism, where separate agents had private information which they summarised truthfully into simple recommendations for the decision maker. More generally, screening was motivated early in a search process, where information acquisition constraints lead to coarse information and it is not yet optimal to use a more accurate but costly procedure involving sequential search or pairwise comparisons.

Chapter Three showed that when choosing between two options, using different thresholds to partition possible realisations of utility minimises the cost of errors in the decision stage. Proposition 3.2 demonstrated that introducing a small bias into a symmetric case leads to a first order gain and only a second order loss. The optimality of introducing bias is expected to hold generally for unimodal continuous distributions of project value. These include those in which interval sets are fuzzy, so there is a chance that realisations close to the threshold lead to the wrong signal, and a broad class of models where the coarse signal is positively correlated with underlying utility or quality, but may involve errors.

Proposition 4.2 showed that optimal fully asymmetric screening (or communication) in the case of N projects is similar to a sequential search without recall, such as the *house selling problem* described by Simon (1955). However, as well as requiring complex calculation to implement, the decision maker must identify every project uniquely and remember the threshold used for every different screen. Therefore Chapter Five investigated partially asymmetric screening, where the decision maker uses an exogenous characteristic to identify projects, and applies the same threshold to screen all projects within a group. It showed that it is optimal to make the screening threshold lower for the minority group, but a project in the majority group, which passes a higher threshold, is chosen in preference to those in the minority group in the decision stage. When the decision maker has a range of characteristics with which to group projects, it is optimal to create a majority and minority rather than two groups of equal sizes. Under optimal partially asymmetric screening, projects in the majority group have a higher chance of being chosen *a priori*.

These intuitions were discussed in relation to the *biased screening* models in the literature on the economics of discrimination. This part of the thesis provided a motivation for discrimination without exogenous economic differences between groups. In addition, it can explain *minority discrimination*, where discrimination is against a minority group specifically, as opposed a separating equilibrium in a search or signalling problem with multiple equilibria, in which the *majority* could just as easily face discrimination. As an example, the intuitions developed in Chapter Five extend to the model of Cornell and Welch (1996), meaning that discrimination would be optimal even without their assumption that the decision maker gets less accurate signals from a minority group. A numerical analysis confirms that under this information structure, biasing the testing process is optimal and a minority group faces a lower chance of being chosen *a priori*.

A promising extension of this analysis would be for the decision maker to choose more than one project, so the screening process occurs early in a search and is followed by a more accurate (but costly) sequential search or pairwise comparison. The utility function at the end of the screening process could be the expected sum of the valuations of the chosen projects, the expected maximum, or an endogenous function depending on the optimal search procedure after the initial screen. Such a heuristic could approximate some suggested models of consumer search. A simple numerical analysis suggests biased screening can be optimal in some of these situations, although a formal analysis would depend on the relative processing costs of screening and search which is beyond the scope of this thesis.

Given that screening models, and specifically bias screening models, are often assumed in the literature of the economics of discrimination, it is surprising that little work has been done to investigate whether this bias could result from optimisation without assuming exogenous differences between the groups. As well as demonstrating the optimality of asymmetry in a class of screening or communication problems, this thesis has shown that discrimination against a minority can arise endogenously as part of an optimal screening process.

Part II

Endogenous Analogy Classes

Chapter 7

Introduction

An analogy-based expectations equilibrium (Jehiel, 2005) involves players bundling nodes at which their opponents move into analogy classes, and forming expectations that the opponents behave in the same way within each class. At every node players choose a best response to their beliefs, and expectations are consistent with the *average* behaviour within an analogy class. This thesis aims to refine this approach, restricting the range of behaviour a player will group in a *robust* analogy class. The underlying motivation for refinement is that players form their analogy classes *endogenously* after observing past histories of the game. The thesis will argue that players are more likely to form analogies over nodes in which the opponent's behaviour is similar, and when suboptimal actions would not be very costly.

While these ideas could be applied to an analogy-based expectations equilibrium of any game, this thesis focuses on a class of games with *finite horizon problems.* These are timing games with complete and perfect information, such as Rosenthal's (1981) Centipede game, in which players take turns to move and receive payoffs which increase over time, until one player *takes*, ending the game.[1] There is a trade off between a player taking first and receiving a larger share of the current total, and passing, which could lead to a larger payout in the future if the opponent also *passes.* McKelvey and Palfrey (1992) show that in an experiment, players do indeed *pass* for some period of time, even though this class of games has a unique subgame-perfect equilibrium, which is solvable by backwards induction, in which players *take* at every node. This would mean that the game ends at the first node and both players receive a low payoff. In the class of games considered, the analogy approach is particularly relevant because the decision faced by the opponent at each node is very similar. More generally, the problem facing a player would be very simple in these games *if* he was aware

[1] Chapter Eight gives a detailed description of the class of games considered. Figures 2.2 and 2.3 provide examples.

of his opponent's strategy. The complexity arises because the opponent's strategy is unknown, and therefore focusing on a method which simplifies beliefs directly, rather than simplifying strategies, seems particularly appropriate.

Chapter Nine will discuss the analogy based expectations equilibrium and apply it to the class of games described in Chapter Eight. It will relate the existence of analogy based-expectations equilibria involving passing to parameters which specify the length and payoff structure of the game. Even when the unique pure strategy equilibrium is that in which players always *take*, some games still have a mixed strategy equilibrium in which passing occurs. The mixed strategy analogy-based expectations equilibria which are developed in Chapter Nine are not unique. In fact there is a continuum of such equilibria, even when players form analogies over just two nodes. Introducing the idea of robustness serves a joint purpose, reducing the set of mixed strategy equilibria and formally capturing the idea that players form their analogy classes *endogenously*.

Chapter Ten will introduce parameters to measure the robustness of analogy classes. It will argue that players are less likely to form analogy classes over nodes in which the opponent's behaviour is very different, and form analogies more carefully when suboptimal actions could prove very costly. These motivations lead to two approaches. Firstly, refinement could be based on restricting the variation of behaviour permitted within an analogy class. This is based on the consistency between a player's analogy-based expectations and the beliefs he would hold if the analogy class was as fine as possible.[2] An alternative approach is to measure the suboptimality of a player's actions resulting from an analogy class. Although these approaches are quite separate (and have different domains) it will be shown that they are closely related and lead to very similar restrictions on behavioural strategies. Applying the refinements to the analogy classes developed in Chapter Nine leads to insights of how the robustness of an analogy class depends on the payoff structure and length of the game. In addition, mixed strategy analogy equilibria are dramatically more robust than pure strategy equilibria involving passing. For example, a pure strategy equilibrium in the Centipede game may need an analogy to be formed over 333 turns to be as robust as a mixed strategy equilibrium formed over just 4 turns. This is because in any analogy-based expectations equilibrium involving only pure strategies, one of the players must observe his opponent taking at a node in which he expected the opponent to *pass* with high probability.[3] As well as being inconsistent with his analogies, in a game such as the Doubling Dollar game, illustrated in Figure 8.3, this leads to a payoff of

[2] These equal the opponent's actual behavioural strategies.
[3] Except in the unique subgame perfect Nash equilibrium.

zero. Although mixed strategy equilibria also involve one of the players taking with certainty at some point, this node is observed with a much lower frequency.

Chapter Eleven will discuss extending the approach developed in Chapter Ten to the case when players have multiple analogy classes. For a given level of sophistication, a repeated game with many nodes has multiple combinations of robust analogy classes, which could be thought as representing different degrees of coordination between the players. For example, the most collusive game involves players passing for a long period, followed by a short period of mixing at the end of the game. The least collusive game is the subgame-perfect equilibrium. The approach proposed in this thesis provides an intuitive solution to the finite horizon problem: an equilibrium consists of players passing for a given number of nodes and then mixing towards the end of the game. As well as placing a bound on the most collusive analogy class for a given level of sophistication, the model provides some restrictions on the structure and size of the analogy classes which can be formed, and the type of behaviour which might be expected within a mixed strategy analogy class in equilibrium. Chapter Eleven will also propose a generalisation of the analogy-based expectation equilibrium concept by incorporating robustness in the requirement for consistency.

In Chapter Twelve a number of other models of bounded rationality have been used to explain why the subgame-perfect equilibrium in finite horizon problems might not arise. Rubinstein (1998) provides a general discussion of the machine game approach to solving finite horizon problems, based on the intuition is that the complexity of a strategy can be represented by the number of states needed to implement it, so boundedly rational players may use simple strategies such as *always pass*. Kreps *et al.* (1982) show that long periods of passing can be sustained in the finitely repeated prisoner's dilemma under incomplete information. This approach can be used to generate passing in the Centipede game when there is a small probability that player 2 is a *cooperative type*. Naturally the approach taken in this thesis closely follows Jehiel (2005), which introduces the analogy-based expectations equilibrium and applies it to a broader range of games. The contribution of this thesis is to endogenise the formation of analogy classes within a specific class of games, and use this to motivate the importance of mixing analogies in finite horizon problems. Other approaches which will be discussed in Chapter Twelve include McKelvey and Palfrey's (1998) quantal response equilibrium, Radner's (1980) notion of perfect ε equilibrium and models of reciprocity (Rabin, 1993) and machine learning (Ponti, 2000).

Chapter 8

A Generalised Centipede Game

The aim of this chapter is to present the class of game that is analysed formally in this thesis. These are finite games of complete and perfect information in which two players $i = 1$, 2 move at alternate nodes, which are indexed by $X_{n,i}$, the node where player i gets to move for the nth time. Player 1 moves in the first node, $X_{1,1}$ and in the subsequent N nodes indexed by $X_{n,1}$. Player 2 moves in the N nodes indexed by $X_{n,2}$, so there is a finite number of nodes in the supergame equal to $2N$. At each node there are two actions available to each player denoted by $a = \{pass,\ take\}$. If a player i *passes* at node $X_{n,i}$ the game continues to the next node in which it is player j's turn to move. If player 1 *takes* at node $X_{n,1}$ the game ends. Player 1 receives a vNM utility payoff of $A_{n,1}$ and player 2 receives a payoff of $B_{n,2}$. If player 2 *takes* at node $X_{n,2}$ the game ends, with players 1 and 2 receiving utilities of $A_{n,2}$ and $B_{n,2}$ respectively. If player 2 *passes* in the final turn $X_{N,2}$ the game ends and players 1 and 2 receive payoffs of $A_{N+1,1}$ and $B_{N+1,1}$ respectively. Such a game is illustrated in Figure 8.1.

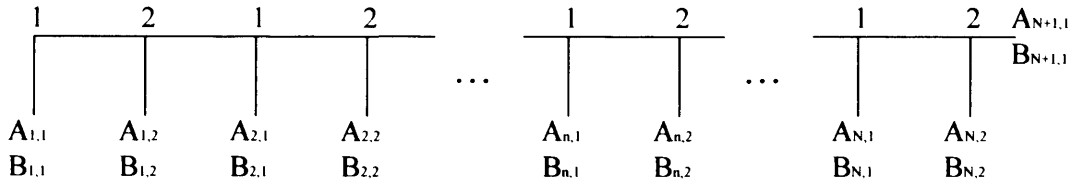


Figure 8.1: A Generalised Centipede Game

It is assumed that payoffs satisfy Equations 8.1 and 8.2 where appropriate:

$$A_{n+1,1} > A_{n,1} > A_{n,2} \geq A_{n-1,2}\ \forall\ n \tag{8.1}$$

$$B_{n+1,2} > B_{n,2} > B_{n+1,1} \geq B_{n,1}\ \forall\ n \tag{8.2}$$

This creates conflicting incentives. As $A_{n+1,1} > A_{n,1}$, player 1 has an incentive to *pass* at node $X_{n,1}$ if he expects player 2 to *pass* at node $X_{n,2}$. If player 1 expects player 2 to *take* at node $X_{n,2}$, then the optimal response is to *take* first in node $X_{n,1}$, preempting player 2, because $A_{n,1} > A_{n,2}$. The same incentives apply for player 2 at nodes in which he moves. Behavioural strategies for the game are defined as follows: at node $X_{n,1}$, the node where player 1 moves for the nth time, he *takes* with probability p_n and *passes* with probability $(1 - p_n)$. At node $X_{n,2}$, the node where player 2 moves for the nth time, he *takes* with probability q_n and *passes* with probability $(1 - q_n)$. Therefore a full behavioural strategy profile for player 1 is defined $p = \{p_1,\ p_2, ...p_n, ...p_N\}$, and for player 2, $q = \{q_1,\ q_{2,}...q_n, ...q_N\}$. The history $h\left(X_{n,1}\right)$ conditional on reaching node $X_{n,1}$ is simply that players have *passed* in all previous nodes.

Despite the conflict in incentives, this class of games has a unique subgame-perfect equilibrium in which players *take* at all nodes, meaning $p = \{1, 1, ...1, ...1\}$ and $q = \{1, 1, ...1, ...1\}$. In this case players receive payoffs of $A_{1,1}$ and $B_{1,1}$ even if the gain from taking is small and the benefit from passing large, as in the Centipede game illustrated in Figure 8.2. As the game has complete and perfect information, it is solvable by backwards induction. In the final node $X_{N,2}$, player 2's strategy is to *take* as $B_{N,2} > B_{N,1}$. Knowledge of player 2's rationality allows player 1 to predict that 2 *takes* at node $X_{N,2}$, and therefore player 1 chooses a best response, to *take* at node $X_{N,1}$. Player 2 knows that player 1 is rational and in addition, that player 1 knows that player 2 himself is rational, so the best response at node $X_{N-1,2}$ is also to *take*. Increasing orders of knowledge of rationality allow players to predict that their opponent *takes* in all future nodes, meaning this procedure can be iterated backwards, generating the unique solution in which players *take* at every node. Such a process corresponds to the iterative deletion of weakly dominated strategies. As well as being the unique subgame-perfect equilibrium, the outcome in which player 1 *takes* in node $X_{1,1}$ and players 1 and 2 receive payoffs of $A_{1,1}$ and $B_{1,1}$ is also the unique Nash equilibrium outcome.

This thesis applies the analogy class approach to a subset of such timing games, where the problem faced by players is similar at every node. The assumption is that payoffs at node $X_{n+1,i}$ are a constant linear transformation of the payoffs at node $X_{n,i}$, so that where appropriate Equations 8.3 and 8.4 hold.[1]

$$A_{n+1,i} = \mu + \lambda A_{n,i} \ \forall\, n\ ,\ i \tag{8.3}$$

$$B_{n+1,i} = \mu + \lambda B_{n,i} \ \forall\, n, i \tag{8.4}$$

[1] Equations 8.1 and 8.2 must still hold. If the game is normalised (see Definition 8.1) this means μ and λ are positive constants with $\mu + \lambda > 1$.

Firstly, note that the parameters μ and λ encompass a large range of timing games. For example, setting $\mu = 1$ and $\lambda = 1$ gives Rosenthal's (1981) specification of the Centipede game which is illustrated in Figure 8.2, and setting $\mu = 0$ and $\lambda = 2$ gives the Doubling Dollar game illustrated in Figure 8.3. Alternative specifications give the games used empirically by McKelvey and Palfrey (1992) and in theoretical learning models by Ponti (2000). Secondly, the analogy class approach is particularly well motivated when the problem faced at different nodes is similar, as it is more reasonable for a player to expect his opponent to behave in the same way. Such payoffs could be motivated if the supergame represents the reduced form of an underlying, finitely repeated game in which payments are accrued continually during the game. Thirdly, by introducing these parameters, it is possible to consider comparative statics for a class of games. λ acts as a scaling effect, determining whether the gain from being the player to *take* increases (if $\lambda > 1$) or decreases (if $\lambda < 1$) as the game progresses. μ has an additive effect, increasing payoffs by the same amount regardless of which player *takes*.[2] This assumption simplifies the concepts underlying the analogy class approach in Centipede-like games. However, the ideas for refinement extend much more broadly, and these will be discussed in Chapter Eleven.

The approach taken in this thesis is to motivate ideas using the Centipede and Doubling Dollar games illustrated in Figures 8.2 and 8.3 respectively. Intuitions about the existence and robustness of equilibria are then extended to games with parameters μ and λ. It is useful to normalise these other games, so that they can be specified fully in terms of μ and λ. Such a normalisation is proposed in Definition 8.1.

Definition 8.1 ***Normalisation:*** *The game illustrated in Figure* 8.1 *in which payoffs satisfy Equations* 8.3 *and* 8.4 *can be normalised by applying the same linear transformation of utility to player i's payoffs at all nodes so* $A_{1,1} = 1$ *and* $A_{1,2} = B_{1,1} = 0$.[3]

The first game considered is the Centipede game (Rosenthal, 1981) in which $\mu = 1$ and $\lambda = 1$. This is illustrated in Figure 8.2 for the case when $N = 4$ (each player moves 4 times so there is a total of 8 nodes).[4]

[2]For example, increasing μ for player 1 will increase the payoffs $A_{n,1}$ and $A_{n,2}$ by the same amount.

[3]Note that normalisation does *not* mean that $B_{1,2} = 1$. This will not affect the analogy equilibrium nor the proposed refinements, and for simplicity, creating a degree of symmetry between the players means that $B_{1,2} = A_{2,1} = \mu + \lambda$.

[4]Note that the Centipede and Doubling Dollar games illustrated in Figures 2.2 and 2.3 respectively are not normalised. To normalise the Centipede game is is necessary to subtract 1 from all of player 2's payoffs $B_{n,i}$. To normalise the Doubling Dollar game all of player 2's payoffs, $B_{n,i}$, must be doubled. Normalisation is not necessary for these games, as it is carried out implicitly in the refinement concept.

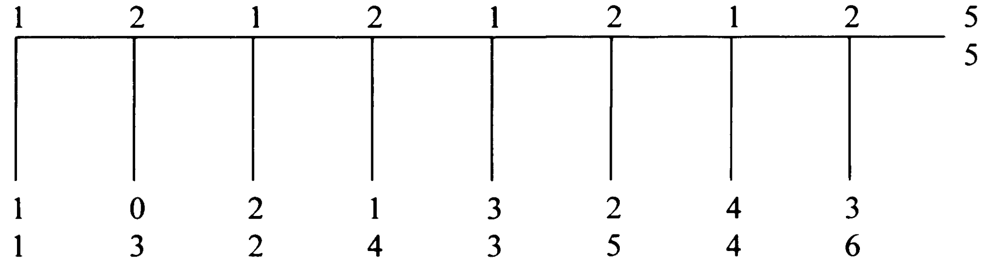


Figure 8.2: The Centipede Game when N=4

The second game considered is the Doubling Dollar game (similar to the Dollar Game proposed in Reny, 1993) where $\mu = 0$ and $\lambda = 2$. This is illustrated in Figure 8.3 for the case when $N = 4$ (each player moves 4 times).

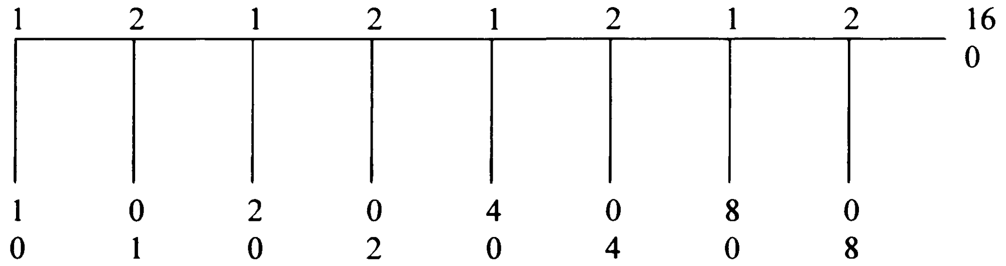


Figure 8.3: The Doubling Dollar Game when N=4

For *any* payoffs satisfying Equations 8.1 and 8.2, behavioural strategies exist which leave the opponent indifferent between mixing and passing at any node, because $A_{n+1,1} > A_{n,1} > A_{n,2}$ implies that $A_{n,1}$ can be expressed as a convex combination of $A_{n+1,1}$ and $A_{n,2}$. In the games considered in the main part of this thesis, which also satisfy Equations 8.3 and 8.4, the utility payoff at node $X_{n+1,i}$ is a constant linear transformation of the payoff at node $X_{n,i}$. This implies that $A_{n,1}$ can be expressed as *the same* convex combination of $A_{n+1,1}$ and $A_{n,2}$, and therefore *the same* behavioural strategy at all nodes for player i leaves player j indifferent at every node. This is demonstrated in Proposition 8.1.

Proposition 8.1 *The assumption that payoffs at node $X_{n+1,i}$ are a constant linear transformation of the payoffs at node $X_{n,i}$,*[5] *whenever applicable, is sufficient but not necessary for the existence of behavioural strategies $p = \{\widehat{p}, \widehat{p}, ..., \widehat{p}\}$ and $q = \{\widehat{q}, \widehat{q}, ..., \widehat{q}\}$ which leave the opponent indifferent between passing and taking at every node (except for player 2 in the final node, $X_{N,2}$).*

[5] In other words, that Equations 2.3 and 2.4 hold.

Proof. Indifference for player 1 at node $X_{N,1}$ implies that

$$\begin{aligned} A_{N,1} &= \widehat{q}A_{N,2} + (1-\widehat{q})A_{N+1,1} \qquad (8.5) \\ &\Rightarrow \widehat{q} = \frac{A_{N+1,1} - A_{N,1}}{A_{N+1,1} - A_{N,2}} \\ &= \frac{(\mu + \lambda A_{N,1}) - (\mu + \lambda A_{N-1,1})}{(\mu + \lambda A_{N,1}) - (\mu + \lambda A_{N-1,1})} = \frac{A_{N,1} - A_{N-1,1}}{A_{N,1} - A_{N-1,1}} \\ A_{N-1,1} &= \widehat{q}A_{N-1,2} + (1-\widehat{q})A_{N,1} \qquad (8.6) \end{aligned}$$

Therefore player 1 is also indifferent at the previous node $X_{N-1,1}$. Repeating this process implies that $A_{n,1} = \widehat{q}A_{n,2} + (1-\widehat{q})A_{n+1,1}$ for all $1 \leq n \leq N$ and by symmetry that $B_{n,2} = \widehat{p}B_{n,1} + (1-\widehat{p})B_{n+1,2}$ for all $1 \leq n \leq N-1$. In the final node $X_{N,2}$, player 2 is never indifferent (as he is not responding to player 1's strategy). Therefore this assumption is sufficient for the existence of $\widehat{q}$ and $\widehat{p}$. That it is not necessary follows from the observation that Equations 8.3 and 8.4 can be satisfied by specifying payoffs to player 1 without the restriction of constant linear transformations. ■

Therefore if player 1 holds the belief that player 2 *takes* with probability $\widehat{q}$ at all nodes, then player 1's indifference at any node implies indifference at all others in which player 1 moves. In the class of timing games satisfying Equations 8.3 and 8.4, $\widehat{q}$ can be expressed in terms of the starting utilities $\widehat{q} = \frac{A_{2,1}-A_{1,1}}{A_{2,1}-A_{1,2}}$, which becomes $\widehat{q} = \frac{\mu+\lambda-1}{\mu+\lambda}$ if the game is normalised. This is a very useful simplification as the constant $\widehat{q}$ contains all the information about player 1's payoffs which is relevant to an equilibrium. The assumption that $\widehat{q}$ is constant (or constant over some range) is necessary for the simple interpretation of analogy classes used formally in this thesis, although Chapter Eleven will demonstrate that the same approach could be generalised to consider other games. Proposition 8.1 leads to Definitions 8.2 and 8.3.

Definition 8.2 *The constant $\widehat{q}$ is defined so that if player 1 believes that player 2* takes *with probability $\widehat{q}$ at all nodes, then player 1 is indifferent between passing and taking at all nodes.*

Definition 8.3 *The constant $\widehat{p}$ is defined so that if player 2 believes that player 1* takes *with probability $\widehat{p}$ at all nodes, then player 2 is indifferent between passing and taking at all nodes.*

The fact that $\widehat{q}$ contains all the information about player 1's payoffs which is relevant to an equilibrium raises the question of why the game is defined in terms of *two* parameters μ and λ. The answer is that these have implications once the *robustness* of an analogy-based expectations equilibrium is considered. For example, both the

Centipede and Doubling Dollar games illustrated in Figure 8.2 and 2.3 respectively have the same $\widehat{p} = \widehat{q} = \frac{1}{2}$, meaning that if each player believes the opponent *takes* at all nodes with probability $\frac{1}{2}$, both players are indifferent at all nodes in which they move. However, the incentive to *take* first is greater in the Doubling Dollar game, and therefore the requirement for an analogy class to be robust is more severe.

It should be noted that the Centipede and Doubling Dollar games satisfy the additional assumption that $\widehat{p} = \widehat{q}$,[6] and it was implicitly assumed that μ and λ are the same in Equations 8.3 and 8.4. These are simplifying assumptions which generate some symmetry between the players.[7] Extending the analysis to games in which μ, λ or $\widehat{p}$ and $\widehat{q}$ are not the same across players is straightforward.

[6] This is equivalent to $\frac{B_{2,2}-B_{1,2}}{B_{2,2}-B_{2,1}} = \frac{A_{2,1}-A_{1,1}}{A_{2,1}-A_{1,2}}$

[7] The term *symmetry* here is used casually, as the game is clearly not symmetric.

Chapter 9

Analogy-Based Expectations Equilibrium

This section describes the concept of an analogy-based expectations equilibrium in the context of the perfect information finite horizon games described in Chapter Eight. Some examples of analogy beliefs are used to motivate the idea of analogy classes, followed by more detailed examples to illustrate how analogy-based expectations equilibria in pure and mixed strategies might overcome the finite horizon paradox. For a more general exposition, the reader is referred to Jehiel (2005), which provides a more general and complete description of the concept and its application to other games. The contribution of this chapter is to extend the analysis to mixed strategy analogy-based expectations equilibria within the class of games described in Chapter Eight, exploring the conditions necessary for such an equilibrium to exist and demonstrating that an equilibrium in mixed strategies may exist even when there is not one in pure strategies.

Each player i forms analogy classes over the nodes $X_{1,j}$, $X_{2,j}$, ..., $X_{n,j}$, ..., $X_{N,j}$ in which *his opponent* moves. Specifically, player 1 forms L analogy classes indexed by Ω_l while player 2 forms M analogy classes indexed by Ψ_m. This chapter follows Jehiel (2005) in assuming that the analogy classes are specified exogenously as a feature of the strategic environment. Chapter Twelve will investigate which analogy classes (and hence equilibria) are robust when players form their analogy classes *endogenously.* Players base their expectations of how an opponent acts on the *average* behaviour in each analogy class and assume that the opponent plays according to this expectation at every node within it. Although player 1's true behavioural strategy profile is $p = \{p_1, ..., p_N\}$, where p_n is the probability that player 1 *takes* at node $X_{n,1}$(the nth time he moves), player 2 forms an expectation that player 1 *takes* with probability $\widetilde{p}_m$

at every node in the analogy class Ψ_m. Likewise player 1 forms an expectation that player 2 *takes* with probability $\tilde{q}_l$ at every node in the analogy class Ω_l. This is best explained using examples.

Example 9.1 *Assume that player* 1 *in the Centipede game illustrated in Figure* 9.1 *has two analogy classes* $\Omega = \{\Omega_1 , \Omega_2\}$ *where* $\Omega_1 = \{X_{1,2} , X_{2,2}\}$ *and* $\Omega_2 = \{X_{3,2} , X_{4,2}\}$. *This means that player* 1 *expects player* 2 *to* take *with probability* $\tilde{q}_1$ *in his first and second moves and to* take *with probability* $\tilde{q}_2$ *in his third and fourth moves. Player* 1*'s analogy-based expectations* $\tilde{q}_1$ *and* $\tilde{q}_2$ *combine to form a perceived strategy profile* $\tilde{q} = \{\tilde{q}_1, \tilde{q}_1, \tilde{q}_2, \tilde{q}_2\}$.

Example 9.2 *Assume that player* 2 *in the Centipede game illustrated in Figure* 9.1 *has a single analogy class* $\Psi = \{X_{1,1}, X_{2,1}, X_{3,1}, X_{4,1}\}$. *This means that player* 2 *expects player* 1 *to* take *with probability* $\tilde{p}_1$ *at every node. Player* 2*'s perceived strategy profile of player* 1*'s behavioural strategy is* $\tilde{p} = \{\tilde{p}_1, \tilde{p}_1, \tilde{p}_1, \tilde{p}_1\}$.

Recall that behavioural strategies were defined for player 1 as $p = \{p_1, ..., p_N\}$ and for player 2 as $q = \{q_1, ..., q_N\}$. The notation is summarised in Table 9.1. Having motivated analogy classes, it is necessary to define two conditions necessary for an analogy-based expectations equilibrium.

Definition 9.1 ***Optimality:*** *It is assumed that both players play sequential best responses to their analogy beliefs. A strategy* $p(\tilde{q})$ *for player* 1 *is a sequential best response to the analogy expectations profile* $\tilde{q}$ *if and only if for all strategies* p', *the expected utility for player* 1 *is weakly maximised so*

$$U_1 \left[p(\tilde{q}) \ , \tilde{q}\right] \geq U_1(p' \ , \tilde{q}) \ \forall \ p'$$

Definition 9.2 ***Consistency:*** *In equilibrium, the profile of analogy-based expectations must be consistent with the average true behaviour in each analogy class. The simplest way to express this consistency requirement is using expectations as shown in Equation 9.1:*

$$\tilde{q}_l = \frac{E\left[\textit{number of times } 1 \textit{ observes } 2 \text{ take } \textit{in } \Omega_l \mid p \ , \ q\right]}{E\left[\textit{number of times } 1 \textit{ observes } 2 \textit{ move in } \Omega_l \mid p \ , \ q\right]} \tag{9.1}$$

Defining $\Pr_{p,q}(X_{n,i})$ as the probability that node $X_{n,i}$ is reached, given strategy profiles p and q, Equation 9.1 can be rewritten as Equation 9.2. An equivalent derivation for $\tilde{p}_m$ gives Equation 9.3. To be consistent, 9.2 and 9.3 must hold for all analogy classes Ω_l and Ψ_m.

$$\widetilde{q}_l = \frac{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})\, q_n}{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})} \tag{9.2}$$

$$\widetilde{p}_m = \frac{\sum_{X_{n,1}\in\Psi_m} \Pr_{p,q}(X_{n,1})\, p_n}{\sum_{X_{n,1}\in\Psi_m} \Pr_{p,q}(X_{n,1})} \tag{9.3}$$

An important feature of the equilibrium is that behaviour within an analogy class is weighted by the frequency with which it is observed. An interpretation of consistency is that a player uses a very long history to forecast behaviour in the analogy class Ω_l, forming an analogy expectation by averaging behaviour at all nodes $X_{n,2} \in \Omega_l$, within it. If this is the result of a process in which players eventually learn to have analogy expectations, then it is not necessary to know an opponent's analogy classes or even payoffs. The notion of strong consistency (Jehiel, 2005) is used here meaning that beliefs off the equilibrium path must also be consistent with the strategy profile.

Definition 9.3 *An assessment* $(p,\ q,\ \widetilde{p},\ \widetilde{q})$ *is an* ***Analogy-Based Expectations Equilibrium*** *if and only if*

1) p and q are sequential best responses to $\widetilde{q}$ and $\widetilde{p}$ respectively
2) $\widetilde{p}$ and $\widetilde{q}$ are strongly consistent with $(p\ ,\ q)$

Jehiel (2005) shows that if every player has the finest possible analogy partitions then the analogy-based expectations equilibrium is also a subgame-perfect equilibrium. He also demonstrates that every finite environment (including a specification of player's analogy classes) has at least one analogy-based expectations equilibrium. Combining the analogy-based expectations equilibrium approach with Proposition 8.1 gives Proposition 9.1.

Proposition 9.1 *If* $\widetilde{p} = \widehat{p}$ *and* $\widetilde{q} = \widehat{q}$ *where* $\widetilde{p}$ *and* $\widetilde{q}$ *are consistent as defined in Definition* 8.3*, then the assessment* $(p,\ q,\ \widetilde{p},\ \widetilde{q}) = (p,\ q,\ \widehat{p},\ \widehat{q})$ *is an analogy-based expectations equilibrium for* ***any*** *game satisfying the requirements in Proposition* 8.1 *with the same* N, Ω, Ψ, $\widehat{p}$ *and* $\widehat{q}$.

Proof. The proof follows from Definition 9.3 of an analogy-based expectations equilibrium and from Proposition 8.1. If the analogy-based expectations $\widehat{p}$ and $\widehat{q}$ are consistent, they are sufficient to make players 1 and 2 indifferent at all nodes (by Definitions 8.2 and 8.3) therefore any behavioural strategies are optimal. ■

Proposition 9.1 follows directly from the way $\widehat{p}$ and $\widehat{q}$ were defined, but it will be useful to refer to it in later analysis. It formalises the discussion following Proposition 8.1, that $\widehat{q}$ and $\widehat{p}$, the beliefs required to make players 1 and 2 indifferent, capture all of the information required about the payoff structure of the game necessary to derive an analogy-based expectations equilibrium in which both players mix. Therefore if $(p', q', \widehat{p}, \widehat{q})$ is an analogy-based expectations equilibrium in the Centipede game in Figure 8.2, then it is also an analogy-based expectations equilibrium of the Doubling Dollar game in Figure 8.3, as in both cases $\widehat{p} = \widehat{q} = \frac{1}{2}$.

9.1 Summary of Notation

$X_{n,i}$	Node at which player i moves for the nth time
$A_{n,i}$	Utility player 1 receives if the game ends with *take* at node $X_{n,i}$
$B_{n,i}$	Utility player 2 receives if the game ends with *take* at node $X_{n,i}$
$p = \{p_1, ..., p_N\}$	Behavioural strategy profile for player 1: *take* with probability p_n at node $X_{n,1}$
$\Omega_l = \{X_{.,2}, ..., X_{.,2}\}$	An analogy class of player 1 grouping nodes at which player 2 moves
$\widetilde{q}_l$	Player 1's analogy-based expectation of player 2's behavioural strategies at nodes in analogy class Ω_l
$q = \{q_1, ..., q_N\}$	Behavioural strategy profile for player 2: *take* with probability q_n at node $X_{n,2}$
$\Psi_m = \{X_{.,1}, ..., X_{.,1}\}$	An analogy class of player 2 grouping nodes at which player 1 moves
$\widetilde{p}_m$	Player 2's analogy-based expectation of player 1's behavioural strategies at nodes in analogy class Ψ_m
$\Pr_{p,q}(X_{n,i})$	Probability that node $X_{n,i}$ is observed given behavioural strategies p and q

Table 9.1: Summary of Notation

9.2 Pure Strategy Analogy Equilibria

This section considers the case when players use pure strategies and have the coarsest possible analogy partitions. A simple example is used to motivate the approach which will then be generalised to a game with N moves for each player.

Example 9.3 *Consider the four period Centipede game illustrated in Figure* 9.1 *and assume that players have the coarsest possible analogy partitions, grouping all nodes at which their opponent moves into a single analogy class.*[1]

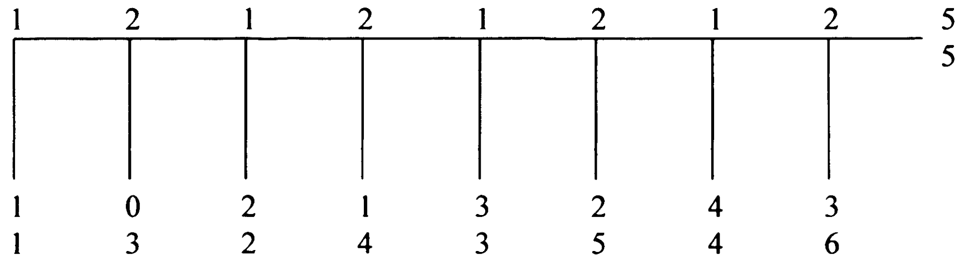


Figure 9.1: The Centipede Game with N=4

There are two possible pure strategy analogy-based expectations equilibria. The first is equivalent to the subgame-perfect equilibrium in which both players take *at all nodes, so that* $p = q = \{1,\ 1,\ 1,\ 1\}$ *and* $\widetilde{q} = \widetilde{p} = 1$.

However, there is a second equilibrium in which player 1 *always* passes *and player* 2 takes *only in the final node so that* $p = \{0,\ 0,\ 0,\ 0\}$ *and* $q = \{0,\ 0,\ 0,\ 1\}$. *Actions at every node are observed equally often with probability* 1, *so for player* 1 *to have consistent analogy expectations it is necessary that* $\widetilde{q} = \frac{0+0+0+1}{4} = \frac{1}{4}$, *while for player* 2, $\widetilde{p} = 0$ *as player* 1 passes *at every node. It remains to verify that each player uses a sequential best response to their analogy beliefs. If player* 1 *believes that* $\widetilde{q} = \frac{1}{4}$ *at all nodes, he will always* pass *as* $\widetilde{q} < \widehat{q} = \frac{1}{2}$. *For player* 2, $q = \{0,\ 0,\ 0,\ 1\}$ *is a best response to the analogy expectation that player* 1 *always* passes.

Therefore the assessment $\left[p = \{0,\ 0,\ 0,\ 0\}\ ,\ q = \{0,\ 0,\ 0,\ 1\}\ ,\ \widetilde{q} = \frac{1}{4}\ ,\ \widetilde{p} = 0\right]$ *constitutes an analogy-based expectations equilibrium in pure strategies when players have the coarsest analogy groupings.*

The unique subgame-perfect equilibrium in which players *take* at every node is also an analogy-based expectations equilibrium supported by analogy beliefs of $\widetilde{q} = (1,\ ...,\ 1)$ and $\widetilde{p} = (1,\ ...,\ 1)$ for *any set of analogy classes.* Example 9.4 shows that a pure strategy analogy-based expectations equilibrium involving passing may also exist,

[1]This was illustrated in Figure 8.2 but is reproduced here to motivate Example 9.3.

providing the game is long enough. This bound is generated by the case when each player *passes* as much as possible and has the coarsest analogy partition, averaging their opponent's behaviour in all nodes. A bound exists because player 2 is rational and so always *takes* in the final decision node, thus $q_N = 1$ puts a lower bound on player 1's consistent analogy expectation that 2 *takes* with probability $\tilde{q}$.

Proposition 9.2 *Assuming $\widehat{p} = \widehat{q}$ and $N \geq 2$, the existence of a passing pure strategy analogy-based expectations equilibrium is bound by $\tilde{q} = \frac{1}{N}$ and therefore $N \geq \frac{\mu+\lambda A_{1,1}-A_{1,2}}{\mu+\lambda A_{1,1}-A_{1,1}}$ so $N \geq \frac{\mu+\lambda}{\mu+\lambda-1}$ in the normalised game.*

Proof. Assume each of the players has the coarsest analogy class and plays equilibrium pure strategies $p = \{0, ..., 0\}$ and $q = \{0, ..., 0, 1\}$ respectively. Every node in equilibrium is visited once meaning player 2 *passes* $(N-1)$ times and *takes* once, so consistent analogy beliefs for player 1 are that player 2 *takes* with probability $\tilde{q} = \frac{1}{N}$. As player 1 always *passes*, consistent analogy beliefs for player 2 are that $\tilde{p} = 0$. To see that $\frac{1}{N}$ is a lower bound on $\tilde{q}$ (or $\tilde{p}$) observe first that if either player *takes* in an earlier node $X_{n,i}$, beliefs will be $\tilde{q} = \frac{1}{n} > \frac{1}{N}$ or $\tilde{p} = \frac{1}{n} \geq \frac{1}{N}$, so the lower bound occurs when *take* is at the end of the game. Secondly, given *take* occurs at the end, any finer analogy partition for player 1 would mean the analogy class containing $q_N = 1$ contains only n nodes, so to be consistent $\tilde{q} = \frac{1}{n}$ is not a lower bound as $\frac{1}{n} > \frac{1}{N}$. Having established that $\tilde{q} = \frac{1}{N}$ creates a lower bound to consistent analogy expectations $\tilde{q}$, it remains to show that this is an equilibrium. From Proposition 9.1 it is optimal for player 1 to *pass* requires that $\tilde{q} = \frac{1}{N} \leq \widehat{q}$ and for player 2 to *pass* providing that $\tilde{p} = 0 \leq \widehat{p}$. This is an equilibrium providing $\tilde{q} \leq \widehat{q} \Longrightarrow \frac{1}{N} \leq \frac{A_{2,1}-A_{1,1}}{A_{2,1}-A_{1,2}} \Longrightarrow N \geq \frac{\mu+\lambda A_{1,1}-A_{1,2}}{\mu+\lambda A_{1,1}-A_{1,1}}$. If the payoffs in the game are normalised as described in Definition 8.1, then $N \geq Max\left(\frac{\mu+\lambda}{\mu+\lambda-1}\ ,\ 2\right)$.
■

For example, in the Doubling Dollar game illustrated in Figure 8.3, player 1 is indifferent at all nodes if player 2 always *takes* with probability $\widehat{q} = \frac{1}{2}$. For *always pass* to be a best response, player 1 must form an analogy-based expectation $\tilde{q} \leq \widehat{q}$. If player 1's analogy beliefs are consistent, it is necessary that $\tilde{q} = \frac{1}{N}$, so combining these gives the condition $\frac{1}{N} \leq \frac{1}{2} \Rightarrow N \geq 2$. Rearranging the final expression in Proposition 9.2 means that a pure strategy analogy-based expectations equilibrium exists providing $\mu + \lambda \geq \frac{N}{N-1}$.

While the strategy profiles $p = \{0, 0, 0, 0\}$ and $q = \{0, 0, 0, 1\}$ might seem reasonable as an equilibrium in the Centipede game, it is less reasonable in the Doubling Dollar game as player 1 always receives a payoff of 0. Although it satisfies the technical requirements for an equilibrium, it seems unsatisfactory when motivated as the result of a learning process. In any pure strategy analogy-based expectations equilibrium involving passing, in every history of the game one of the players observes his opponent *take* with probability 1 at a node in which he expected the opponent to *pass* with high probability. In Chapter Ten it will be argued that mixed strategy analogy classes are dramatically more robust, as taking is closer to mixing (than passing) and pure *take* may be observed with a very low frequency. In addition, mixing behaviour might be more complex to determine, motivating the formation of analogy classes.

9.3 Mixed Strategy Analogy Equilibria

For simplicity this section begins by focusing on the case when players have the coarsest possible analogy partitions and form analogy expectations $\widetilde{q}$ or $\widetilde{p}$ at all nodes at which their opponent moves. The extension of this approach to the case when there are multiple analogy classes will be discussed in Chapter Eleven. This section outlines the conditions required for a mixed strategy analogy-based expectations equilibrium and illustrates the idea in games in which $N = 2$, demonstrating that even in this case, there are multiple mixed strategy analogy equilibria.

This section examines mixed strategy equilibria in which both players mix.[2] If p and q are equilibrium mixed strategy profiles for players 1 and 2 respectively, following Proposition 8.1 mixing is optimal if players are indifferent, so it is necessary that $\widetilde{q} = \widehat{q}$ for player 1 and $\widetilde{p} = \widehat{p}$ for player 2. Recalling that $\Pr_{p,q}(X_{n,i})$ is the probability that node $X_{n,i}$ is observed, for analogy expectations to be consistent is it necessary that $\widetilde{q} = \frac{\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2}) \cdot q_n}{\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2})}$ and $\widetilde{p} = \frac{\sum_{X_{n,1}} \Pr_{p,q}(X_{n,1}) \cdot p_n}{\sum_{X_{n,1}} \Pr_{p,q}(X_{n,1})}$ which can be rewritten as Equations 9.4 and 9.5.[3]

$$\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2})(q_n - \widetilde{q}) = 0 \tag{9.4}$$

$$\sum_{X_{m,1}} \Pr_{p,q}(X_{n,1})(p_n - \widetilde{p}) = 0 \tag{9.5}$$

[2]There are other equilibria in which only one player mixes with sufficiently low probability and the opponent plays a pure strategy, $p = (0, 1)$ or $q = (0, 1)$.

[3]These were derived in Equations 9.2 and 9.3.

Example 9.4 *The optimality and consistency requirements for a mixed strategy analogy-based expectations equilibrium are demonstrated for the Centipede game when $N = 2$ (illustrated in Figure 9.2).*

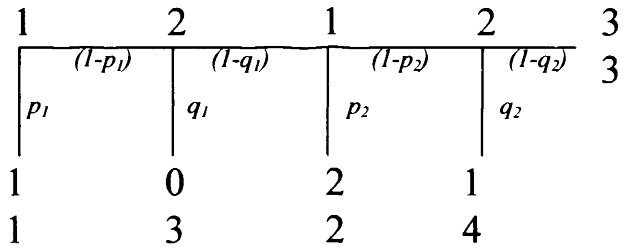


Figure 9.2: The Two Period Centipede Game

For player 1 *to be indifferent at all nodes, he must expect player* 2 *to* take *with probability* $\widehat{q} = \frac{A_{2,1}-A_{1,1}}{A_{2,1}-A_{1,2}} = \frac{2-1}{2-0} = \frac{1}{2}$ *and likewise for player* 2 *to be indifferent at all nodes, he must expect player* 1 *to* take *with probability* $\widehat{p} = \frac{1}{2}$. *For mixing to be optimal, the requirements are that analogy expectations* $\widetilde{q} = \widehat{q} = \frac{1}{2}$, $\widetilde{p} = \widehat{p} = \frac{1}{2}$ *and that* $q_N = 1$.

For analogy expectations to be consistent it follows from Equations 9.4 *and* 9.5 *that*

$$\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2})(q_n - \widetilde{q}) = 0 \Rightarrow (1-p_1)(q_1-\widetilde{q}) + (1-p_1)(1-q_1)(1-p_2)(q_2-\widetilde{q}) = 0$$

$$\sum_{X,1} \Pr_{p,q}(X_{n,1})(p_n - \widetilde{p}) = 0 \Rightarrow (p_1-\widetilde{p}) + (1-p_1)(1-q_1)(p_2-\widetilde{p}) = 0$$

This gives five expressions which can be solved simultaneously:

Optimality for player 1	$\widetilde{q} = \widehat{q} = \frac{1}{2}$
Optimality for player 2	$\widetilde{p} = \widehat{p} = \frac{1}{2}$
Optimality for player 2	$q_2 = 1$
Consistency of 1*'s beliefs*	$(q_1-\widetilde{q}) + (1-q_1)(1-p_2)(q_2-\widetilde{q}) = 0$
Consistency of 2*'s beliefs*	$(p_n-\widetilde{p}) + (1-p_1)(1-q_1)(p_2-\widetilde{p}) = 0$

These can be expressed in terms of p_1 *as Table* 9.2 *shows.*

p_1	q_1	p_2	q_2	$\widetilde{p}$	$\widetilde{q}$
p_1	$1-\frac{1}{3(1-p_1)}$	$2-3p_1$	1	$\frac{1}{2}$	$\frac{1}{2}$

Table 9.2: Equilibrium Behavioural Strategies

Thus even when each player forms coarse analogy classes over just two nodes, there are multiple mixed strategy analogy-based expectations equilibria. This follows intuitively from Example 9.4 as the equilibrium imposes five constraints over six parameters p_1, p_2, q_1, q_2, $\widetilde{p}$ and $\widetilde{q}$. In longer games, where players form analogy classes over N nodes, there are still only five constraints, leaving $2N-3$ free parameters. However, generally these restrictions still place bounds on possible mixing behaviour.

Proposition 9.3 *Assuming $\widehat{p} = \widehat{q}$ and $N \geq 2$ the existence of a mixed strategy analogy-based expectations equilibrium is bound by $\widehat{q} = \frac{1-p_N}{N-p_N}$ and therefore $N \geq \frac{1}{1-\left(\frac{A_{1,1}-A_{1,2}}{\mu+\lambda A_{1,1}-A_{1,2}}\right)^2}$ so $N \geq \frac{(\mu+\lambda)^2}{(\mu+\lambda)^2-1}$ in the normalised game.*

Proof. The proof that the bound occurs when players *pass* until the end of the game and have the coarsest analogy classes is equivalent to that in Proposition 9.2. A mixed strategy equilibrium could exist in which player 1 mixes in the penultimate node, so that $p = \{0, ..., 0, p_N\}$ and $q = \{0, ..., 0, 1\}$. For mixing to be a consistent best response for player 1 it is necessary that $\widehat{q} = \widetilde{q}$, and for player 1 to have consistent beliefs the requirement is that $\widetilde{q} = \frac{1-p_N}{N-p_N}$. For player 2 to *pass* until the final node $X_{N,2}$ it is necessary that $\widehat{p} \geq \widetilde{p}$ and consistent expectations require that $\widetilde{p} = \frac{p_N}{N}$. As $\widehat{p} = \widehat{q}$ by assumption, for these conditions to be satisfied $\frac{1-p_N}{N-p_N} = \widetilde{q} = \widehat{q} = \widehat{p} \geq \widetilde{p} = \frac{p_N}{N}$ in equilibrium. Therefore a mixed strategy analogy-based expectations equilibrium can exist in which $p_N = \frac{1-N\widehat{q}}{1-\widehat{q}}$ providing $\frac{1-p_N}{N-p_N} \geq \frac{p_N}{N}$, so substituting and solving the quadratic means $\widehat{q} \geq 1 - \sqrt{\frac{N-1}{N}}$ therefore $N \geq \frac{1}{1-(1-\widehat{q})^2} = \frac{1}{1-\left(\frac{A_{1,1}-A_{1,2}}{\mu+\lambda A_{1,1}-A_{1,2}}\right)^2}$. If the game is normalised as described in Definition 8.1 then $N \geq Max\left(\frac{(\mu+\lambda)^2}{(\mu+\lambda)^2-1}, 2\right)$. ■

Proposition 9.3 implies that if

$$\frac{1}{1-\frac{A_{1,1}-A_{1,2}}{\mu+\lambda A_{1,1}-A_{1,2}}} \geq N \geq \frac{1}{1-\left(\frac{A_{1,1}-A_{1,2}}{\mu+\lambda A_{1,1}-A_{1,2}}\right)^2}$$

then a mixed strategy analogy-based expectations equilibrium exists which involves passing, even though the unique pure strategy analogy based expectations equilibrium outcome involves players taking at every node. If the game is normalised this domain can be expressed $\frac{(\mu+\lambda)}{(\mu+\lambda)-1} \geq N \geq \frac{(\mu+\lambda)^2}{(\mu+\lambda)^2-1}$. In terms of μ and λ this occurs when

$$\frac{N}{N-1} \geq \mu + \lambda \geq \sqrt{\frac{N}{N-1}}$$

For example, if $\lambda = \frac{3}{2}$, $\mu = 0$ and $N = 2$, Proposition 9.3 implies that a mixed strategy analogy based expectations equilibrium exists in which $\widehat{q} = \widehat{p} = \frac{1}{3}$, and $\widehat{q} = \frac{1-p_N}{N-p_N}$ means $p_N = \frac{1-N\widehat{q}}{1-\widehat{q}} = \frac{1}{2}$. Therefore $p = \left\{0\ ,\ \frac{1}{2}\right\}$ and $q = \{0\ ,\ 1\}$ is a mixed strategy analogy-based expectations equilibrium, although none exists in pure strategies.

9.4 Discussion

In the Centipede game in Figure 9.2, there is a mixed strategy analogy-based expectations equilibrium in which $p = \{0.4167\ ,\ 0.75\}$ and $q = \{0.4286\ ,\ 1\}$.[4] This addresses some of the problems with the passing pure strategy equilibrium which were discussed in Section 9.2. Pure *take* is now only observed with probability $\frac{1}{12}$ rather than 1. Player 2's behaviour at q_2' is now to take with probability $\frac{3}{7}$ rather than 0, which is closer to player 1's analogy-based expectation that he mixes with probability $\frac{1}{2}$. In the Doubling Dollar game, player 1 would receive a payoff of 0.9167 which is closer to 1, the amount he would expect under his analogy beliefs. In games in which $N > 2$ these arguments become more powerful.

The next chapter investigates the case when players form analogy classes endogenously. The motivation for this approach is that players would not form analogies over nodes in which the opponent's behaviour is very different, or that they form analogies more carefully when suboptimal actions could prove very costly. It will propose formal measures of the inconsistency of analogy beliefs (with true behaviour) and the suboptimality of a player's behaviour resulting from a poor analogy. These measures allow an investigation of how the payoff parameters, μ and λ, and the length of the game, N, affect the robustness of the different pure and mixed strategy equilibria that have been developed in this chapter.

[4] This follows by substituting $p_1 = \frac{5}{12} = 0.1467$ into Table 9.2.

Chapter 10

Endogenous Analogy Classes

Chapter Nine showed that as well as pure strategy analogy-based expectations equilibria, there is a continuum of mixed strategy equilibria, even when players form analogy classes grouping just two nodes. Rather than defining analogy classes exogenously as part of the strategic environment, this chapter will argue that players form analogies endogenously. Earlier it was suggested that players learn to form consistent expectations after observing a history of past games. This chapter will extend the idea to suggest that players also form their analogy classes from this history.

It is convenient to refer to this process as refinement, whereby players form analogy classes and reject undesirable ones, based either on inconsistency between analogy-based expectations and true behaviour, or the suboptimality of actions responding to a single analogy class. However, the underlying motivation is that players are less likely to form analogy classes over nodes in which the opponent's behaviour is very different, or that they form analogies more carefully when suboptimal actions could prove very costly. Any refinement takes place in players' minds as part of the process by which they form their analogy expectations.

The discussion is formalised by defining measures to analyse the extent of suboptimality and inconsistency, providing a framework for assessing the robustness of an analogy-based expectations equilibrium. To motivate the need for refinement even of mixed strategies, consider the strategy profiles $p = \left\{\frac{2}{3}\ ,\ 0\right\}$ and $q = \{0\ ,\ 1\}$ in Table 10.1. These form a mixed strategy analogy-based expectations equilibrium of the Centipede game in Example 9.4 when players have the coarsest possible analogy partitions, so $\widetilde{p}=\widetilde{q}=\widehat{p}=\widehat{q}=\frac{1}{2}$.[1]

	p_1	q_1	p_2	q_2
Probability of *take*	$\frac{2}{3}$	0	0	1
Deviance from $\widetilde{p}$ or $\widetilde{q}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Probability behaviour is observed	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 10.1: Example of a Mixed Strategy Equilibrium

This analogy-based expectations equilibrium seems unsatisfactory even for unsophisticated players in such a short game. Firstly, it seems unreasonable for the players to fail to recognise that their analogy beliefs differ from the opponent's true behaviour, when this difference is so significant and regularly observed. Secondly, player 1's strategy profile is very suboptimal given player 2's true behaviour.[2] The first criticism questions how inconsistent analogy-based expectations are with the opponent's actual behaviour, while the second investigates how much the player stands to gain by switching to the true optimal strategy. In both cases, the comparison to a benchmark case where the player breaks an analogy class into the finest possible partitions seems the most natural.[3] These motivations for refining analogy-based expectations equilibria are closely related, as optimal beliefs lead to optimal behaviour. It is assumed that to be robust an analogy class must survive both optimality and consistency based refinements.

To illustrate these refinements, they are first applied to simple games in which each player moves twice and holds the coarsest possible analogy partition. This will then be extended to when the game has more nodes but players continue to form coarse analogy classes. In addition, it will argue that some mixed strategy equilibria are unlikely to arise, and permit a formal comparison of mixed and pure strategy analogy-based expectations equilibria. Mixed strategy analogy-based expectations equilibria may be dramatically more robust than the pure strategy equilibria which involve passing. For example, a pure strategy equilibrium in the Centipede game may need the analogy to be formed over 333 turns to be as robust as a mixed strategy equilibrium formed over just 4 turns. Chapter Eleven will discuss the case when players may form multiple analogy classes.

[1] This follows from setting $p_1=\frac{2}{3}$ in Table 9.1.

[2] In either the Centipede or Doubling Dollar game player 1 could double his payoff by setting $p_1=0$, $p_2=1$.

[3] Firstly, if this were not the case refinement would become dependent on which analogy classes an equilibrium was divided to. Secondly, two analogy classes are almost always sufficient to support the true optimal strategy anyway.

Refinement Based on Consistency

This approach considers how much a player's analogy-based expectations differ from his opponent's true behaviour. A player is less likely to form an analogy that the opponent behaves in a similar way at different nodes, if the opponent's true behaviour at those nodes is very different. The *mean absolute deviation* of behaviour within an analogy class is used to measure this difference rather than the variance. Mean absolute deviation puts equal weight on outliers, which are typical in such equilibria (as at some point one of the players must *take* with probability 1) while the variance[4] weights outlying observations more heavily. Despite it being intuitive - *the average deviation from the mean* - mean absolute deviation is rarely used in statistics because of the difficulty of using absolute values. In this case however, it is straightforward to calculate and is similar to the requirement for analogy beliefs to be consistent, because of the statistical result that the average deviation about the mean is zero (when deviations are not absolute).

Definition 10.1 ***A consistency based refinement measure*** $\boldsymbol{\theta_{i,k}}$ *for player i in analogy class k is defined* $\boldsymbol{\theta_{1,l}(q\ ,\ p) = MAD(q)}$ *for player* 1 *and* $\boldsymbol{\theta_{2,m}(\mathbf{p}\ ,\ \mathbf{q}) = \mathbf{MAD}(\mathbf{p})}$ *for player* 2.[5]

The mean behaviour in this case is the observed mean, so it is correctly calculated based on the probability with which behaviour at each node is observed within an analogy class. This corresponds to the analogy expectations for each analogy class $\tilde{p}$ and $\tilde{q}$ respectively. The absolute deviations from this behaviour should also be weighted by frequency, which means that more extreme absolute deviations from $\tilde{p}$ are possible in a robust analogy class the less frequently they are observed. The mean absolute deviations can be expressed:

$$\begin{aligned} \theta_{1,l}(q\ ,\ p) &= \frac{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})\,|\tilde{q}_l - q_n|}{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})} \\ \theta_{2,m}(p\ ,\ q) &= \frac{\sum_{X_{n,1}\in\Psi_m} \Pr_{p,q}(X_{n,1})\,|\tilde{p}_m - p_n|}{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})} \end{aligned}$$

The approach of using a measure of dispersion to form optimal categories is used by Fryer and Jackson (2004) who show that it may be optimal to partition larger groups more finely than smaller groups. However, the game theoretic situation used here is significantly different from their model of a single decision problem.

[4]Instead of using the variance itself, a related dimensionless measure such as the coefficient of variation would be an appropriate alternative to the mean absolute deviation.

[5]As discussed in the text, $MAD(q)$ is the mean absolute deviation of q.

Refinement Based on Optimality

A second approach is motivated by the idea that players form analogies more carefully when suboptimal actions could prove very costly. The greater the relative gain in utility from refining analogy-based expectations and responding optimally to the opponent's true behaviour, the more likely a player is to refine his analogy beliefs and hence the less robust an analogy class. The optimal and expected equilibrium payoffs player 1 receives are calculated based on the opponent's true behaviour (given the analogy beliefs the equilibrium strategy is already optimal). It is not implied that players actually calculate these, as they would need to find the optimal strategy to do so.

The motivation behind this approach is related to that of ε-equilibrium (Radner, 1980), where players have a cost of finding or switching to an optimal strategy, and so only adjust their behaviour if an alternative strategy increases utility by more than ε. However, Radner (1980) uses the approach to increase the set of equilibria from the unique subgame-perfect equilibrium, while here it is used to reduce the set of equilibria in an approach that already contains elements of bounded rationality.

The true expected utility for player i is defined as $U_1[p\ ,\ q]$ when players play according to strategy profiles p and q. The optimal response to an opponent's true strategy is defined $p^*(q)$ and $q^*(p)$ for players 1 and 2 respectively. Proposition 10.1 shows that it is sufficient to examine only pure strategies to find the optimal response to an opponent's strategy profile.

Proposition 10.1 *When players have the finest possible analogy groupings, weakly optimal profiles of pure strategies exist for players* 1 *and* 2, *and are defined as* $p^*(q)$ *and* $q^*(p)$ *respectively.*

Proof. Utility payoffs are a linear function of strategies, so the payoff from a mixed strategy is a convex combination of pure strategy payoffs. See Appendix B.3 for an example of this in an analogy-based expectations equilibrium. ■

An intuitive measure of the suboptimality of a player's actions might be the percentage gain from switching from the equilibrium strategy to the optimal strategy, $\frac{U_1[p^*(q)\ ,\ q]-U_1[p\ ,\ q]}{U_1[p\ ,\ q]}$ for player 1 and $\frac{U_2[p\ ,\ q^*(p)]-U_2[p\ ,\ q]}{U_2[p\ ,\ q]}$ for player 2. This has the advantage of being invariant to the scale of utility, allowing comparisons of different payoff structures and games of different lengths. However, some normalisation is necessary to make these measures invariant to the level of utility.[6] $A_{1,2}$ is subtracted from all

[6]So that adding 100 to all payoffs does not make an analogy equilibrium more robust to payoff based refinement.

of player 1's utilities and $B_{1,1}$ from all of player 2's. This is equivalent to normalising the payoffs in the manner described in Definition 8.1.

Definition 10.2 ***An optimality based refinement measure*** *is defined as $t_{i,k}\,(p\ ,\ q)$ for player i in analogy class k and represents the expected (normalised) percentage gain a player would make by switching to the optimal strategy.*[7]

$$
\begin{aligned}
t_1(p\ ,\ q) &= \frac{U_1\,[p^*(q)\ ,\ q] - U_1\,[p\ ,\ q]}{U_1\,[p\ ,\ q] - A_{1,2}} \\
t_2(p\ ,\ q) &= \frac{U_2\,[p\ ,\ q^*(p)] - U_2\,[p\ ,\ q]}{U_2\,[p\ ,\ q] - B_{1,1}}
\end{aligned}
$$

For example, if the game is normalised then an optimality based refinement measure of $t_1 = 0.1$ means that switching from the equilibrium strategy to the real optimal strategy increases expected utility by 10%.

As well as allowing the measures to be compared between games of different lengths and payoff structures, normalisation provides a comparison of the robustness of an equilibrium between players. This judgement is necessary to determine the most robust analogy-based expectations equilibrium, as a bound exists at which it is not possible to improve the robustness of an analogy class for one player without simultaneously making the opponent's analogy class less robust.

Comparing the Approaches to Refinement

In Example 9.4, each player was indifferent if their opponent passed with a probability of $\widehat{p} = \widehat{q} = \frac{1}{2}$. An equilibrium of the Doubling Dollar and Centipede games has the same consistency based refinement measure as only $\widehat{q}$ and $\widehat{p}$, and not λ and μ individually, enter the equilibrium[8] and robustness measures. Therefore if the Doubling Dollar game is more sensitive to analogy beliefs than the Centipede game, this must be implemented exogenously. However, the consistency based measure is natural in an environment where it is uncertainty about the players' actions that generates the complexity motivating an analogy-based expectations equilibrium.

An optimality based refinement measure has the advantage that it makes sensitivity to the payoff structure endogenous. An implication of this is that if a player plays an optimal strategy, the analogy class is totally robust even if the opponent's behaviour within it is very varied. For example, the mixed strategy analogy-based expectations equilibrium in Table 10.1 is:

[7] All expectations here are based on players having the finest possible analogy partitions. Under analogy-based expectations the current strategy would already be optimal.

[8] As in Proposition 9.1.

Probability of *take*	p_1	q_1	p_2	q_2
	0.6667	0	0	1

In this case $t_2 = 0$ because backwards induction shows that player 2's response to player 1's actions is optimal already, so $U_1 [p^*(q) , q] = U_1 [p , q]$. Despite this, it seems unreasonable that no degree of sophistication allows player 2 to understand that player 1's behaviour at p_1 and p_2 is different.

As well as comparing the robustness of different analogy classes, the optimality and consistency refinement measures could be used to give bounds, reversing the question to ask what equilibria survive refinement for a given level of player sensitivity. For example, an analogy class is considered robust providing $\theta_1 \leq \overline{\theta_1}$ and $t_1 \leq \overline{t_1}$. In the limit, as the equilibrium strategies tend towards the subgame-perfect equilibrium, $\theta_1 = \theta_2 = t_1 = t_2 = 0$.[9] These two approaches are related because when the consistency based refinement is small, a player's analogy-based expectations are close to the opponent's true actions, so the optimal response to the analogy class is close to the fully optimal response. It will be demonstrated in the next section that when players have the coarsest analogy classes and expect the opponent to mix, the most robust equilibria given optimality and consistency based refinement are extremely similar. Even if they do not give similar restrictions, the measures are complementary, as a robust analogy class should survive both types of refinement.

10.1 Refinement of Pure Strategy Analogy Equilibria

This section analyses optimality and consistency based robustness measures of pure strategy analogy-based expectations equilibria in which players *pass*, for a comparison with the robustness of mixed strategy analogy-based expectations equilibria in Sections 10.2 and 10.3. The case when each player has a single analogy class is relatively simple to analyse, as all nodes are observed with probability 1. Player 1 always *passes* while player 2 *passes* until the node $X_{N,2}$ at which he *takes*. The analogy class held by player 2, that player 1 always *passes*, is correct and is therefore totally robust. That held by player 1 is suboptimal however, as the variation in player 2's behaviour means that player 1 is better off taking rather than passing in his own final move.

Proposition 10.2 *If each player has a single analogy class, a pure strategy analogy-based expectations equilibrium involving passing in the Centipede game is robust for any tolerance level if the game is sufficiently long, as $\theta_1 \rightarrow 0$ and $t_1 \rightarrow 0$ as $N \rightarrow \infty$.*[10]

[9] As the subgame perfect equilibrium is reached, t_2 and θ_2 are undefined.

[10] As player 1 always *passes* $\theta_2 = t_2 = 0$.

Proof. In this equilibrium player 1 always *passes* and player 2 *passes* until the last node at which he *takes* with probability 1. As actions at all nodes are observed with equal probability 1, player 1 forms consistent analogy beliefs $\widetilde{q} = \frac{1}{N}$. Therefore the mean absolute deviation within the analogy class player 1 holds about player 2's actions is $\theta_1 = \frac{|0-\frac{1}{N}|(N-1)+|1-\frac{1}{N}|}{N} = 2\frac{N-1}{N^2}$. The payoff received by player 1 in equilibrium is $N-1$ but if he switched to the optimal strategy of taking just before his opponent, in his own final node, he would receive a payoff of N. Therefore $t_1 = \frac{U_1[p^*(q)\,,\,q]-U_1[p\,,\,q]}{U_1[p\,,\,q]-A_{1,2}} = \frac{N-(N-1)}{(N-1)-0} = \frac{1}{N-1}$. As $N \to \infty$, $\theta_1 \to 0$ and $t_1 \to 0$ as required. ■

The measures of robustness θ_1 and t_1 given N for a pure strategy analogy-based expectations equilibrium involving passing are illustrated in Figures 10.1 and 10.2 respectively.[11]

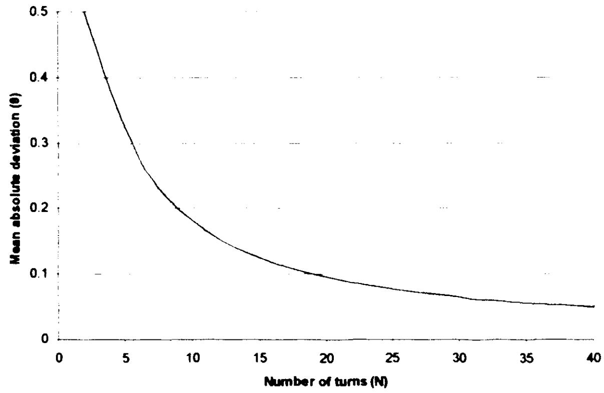


Figure 10.1: Consistency based robustness of a passing pure strategy equilibrium of the Centipede game when each player moves N times

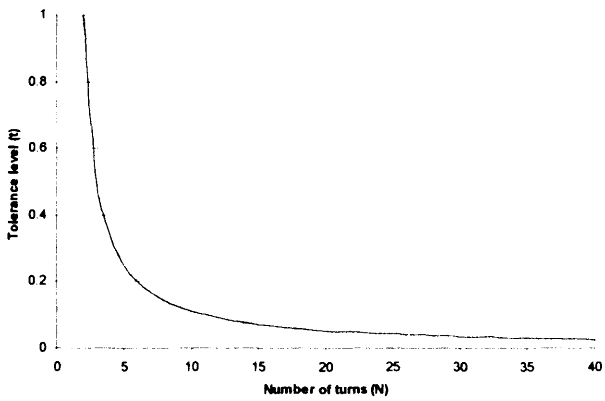


Figure 10.2: Optimality based robustness of a passing pure strategy equilibrium of the Centipede game when each player moves N times

[11]These graphs show continuous approximations as N is an integer.

The first relationship occurs because the absolute deviation[12] is being averaged over an increasingly large number of nodes as $N \to \infty$, and the second because payoffs to both players are increasing, so as a percentage, the gain from switching to the optimal strategy decreases. Therefore the optimality based and consistency based robustness measures follow a very similar trend. However, although both θ_1 or t_1 tend to 0 as N increases, even small values may not be robust when player 2's strategy is so straightforward for player 1 to understand and respond optimally to.

Proposition 10.3 *If each player has a single analogy class, a pure strategy analogy-based expectations equilibrium involving passing in the Doubling Dollar game is not robust for any tolerance level.*

Proof. In the Doubling Dollar game, consistency based refinement puts the same bound on the number of turns as the Centipede game, $\theta_1 = 2\frac{N-1}{N^2}$ derived in Proposition 10.2. However, the optimality based refinement for player 1 is never robust. The payoff received in equilibrium is 0 as player 2 always *takes*, but player 1 could receive a payoff 2^{N-1} by switching to the optimal strategy, and *take* in his final move. Therefore $t_1 = \frac{U_1[p^*(q)\,,\,q]-U_1[p\,,\,q]}{U_1[p\,,\,q]-A_{1,2}} = \frac{2^{N-1}-0}{0-0} = \frac{2^{N-1}}{0}$, and the pure strategy equilibrium involving passing is not expected to be robust in Doubling Dollar games of any length. ∎

The optimality based refinement measure for the Doubling Dollar game is very different from the Centipede game (see Proposition 10.2) and the consistency based refinement measure. This reflects that payoffs in the Doubling Dollar game are more sensitive to the opponent's actions. If the passing analogy-based expectations equilibrium in which each player has a single analogy class is not robust, then no analogy-based expectations equilibrium in pure strategies is robust other than the equivalent of the subgame-perfect equilibrium.[13]

Having observed that there are differences between the Centipede and Doubling Dollar games, an interesting question is how the robustness of a pure strategy equilibrium depends on the payoff structure of the game. The mean absolute deviation of beliefs is independent of the payoffs, and so providing a pure analogy-based expectations equilibrium exists, the consistency based robustness measure is the same (as explained above for the Centipede and Doubling Dollar games). The same is not true for the optimality based refinement.

[12] When players use pure strategies, the absolute deviation equals $2\frac{N-1}{N}$ so is increasing in N, but the effect of averaging it across all nodes dominates this.

[13] Even if the analogy classes were finer, meaning that the final *take* would occur in an analogy class with n nodes, $t_1 = \frac{2^{n-1}}{0}$ would still not be robust.

Proposition 10.4 *For a normalised game in which each player has the coarsest analogy partition, the robustness of the pure strategy passing equilibrium is decreasing in λ and increasing in μ.*

Proof. Assuming for simplicity that the game is normalised so that $A_{1,1} = 1$ and $A_{1,2} = 0$, the payoff to player 1 at node $X_{N,1}$ can be calculated as $\frac{\mu(1-\lambda^{N-1})}{1-\lambda} + \lambda^{N-1}$ and at node $X_{N,2}$ is $\frac{\mu(1-\lambda^{N-1})}{1-\lambda}$. Therefore:

$$t_1 = \frac{\frac{\mu(1-\lambda^{N-1})}{1-\lambda} + \lambda^{N-1} - \frac{\mu(1-\lambda^{N-1})}{1-\lambda}}{\frac{\mu(1-\lambda^{N-1})}{1-\lambda}} = \frac{\lambda^{N-1}(1-\lambda)}{\mu(1-\lambda^{N-1})}$$

$$\frac{dt_1}{d\mu} = -\frac{\lambda^{N-1}(1-\lambda)}{\mu^2\left(1-\lambda^{N-1}\right)} < 0$$

$$\frac{dt_1}{d\lambda} = \frac{\lambda^{N-2}\left[N-1-N\lambda+\lambda^N\right]}{\mu(1-\lambda^{N-1})^2} > 0$$

as $N-1-N\lambda+\lambda^N > 0$ follows from Bernoulli's inequality. Recall that $t_2 = 0$ as player 2's payoff is already optimal. ■

Proposition 10.4 shows that the robustness of the passing equilibrium to optimality based refinement decreases in λ and increases in μ.[14] This is because a larger λ increases the scaling of the difference between taking now and the opponent taking next node all through the game, raising $U_1[p^*(q)\ ,\ q] - U_1[p\ ,\ q]$ and therefore increasing t_1 (reducing robustness). Increasing μ scales up the expected payoff $U_1[p\ ,\ q]$ but not the difference $U_1[p^*(q)\ ,\ q] - U_1[p\ ,\ q]$, so the percentage gain of optimal over expected utility is reduced and the analogy class becomes more robust.

10.2 Refinement of Mixed Strategy Analogy Equilibria in Two Period Games

This section calculates consistency and optimality based robustness measures for mixed strategy analogy-based expectations equilibria. There are three main reasons to expect analogy classes (and therefore equilibria) involving mixing to be more robust than those in which players use only pure strategies. Firstly, the change from mixing to taking is less extreme than the change from passing to taking. Secondly, although any equilibrium involves one player taking with probability 1 when they were expected to mix, this node is observed less often than in a pure strategy equilibrium. For example, if players have the coarsest analogy partitions in the Centipede game in Figure 8.2

[14] For a robust equilibrium the consistency based and payoff based refinement measures should be as small as possible.

when $N = 4$, a mixed strategy equilibrium exists in which pure *take* is observed with less than 1% probability. Thirdly, mixing behaviour may be more complex to analyse than pure strategies.

This section focuses on the Centipede and Doubling Dollar games illustrated in Figures 10.3 and 10.4 respectively, where each player forms analogy classes over the two nodes at which his opponent moves. The later sections will examine the most robust mixed strategy equilibrium when the game has N nodes.

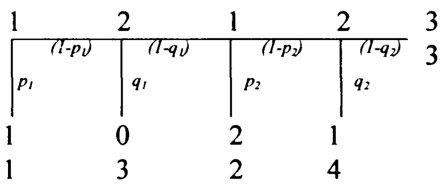


Figure 10.3: Two Period Centipede Game

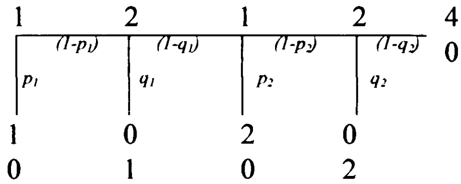


Figure 10.4: Two Period Doubling Dollar Game

It is assumed for the rest of this section that the strategy profiles $p = (p_1 \ , \ p_2)$ and $q = (q_1 \ , \ q_2)$, which generate analogy beliefs $\tilde{p}$ and $\tilde{q}$ respectively, form an analogy-based expectations equilibrium. Following Example 9.2, these equilibria have one free parameter and so behaviour at all nodes can be specified in terms of p_1:[15]

p_1	q_1	p_2	q_2	$\tilde{p}$	$\tilde{q}$
p_1	$1 - \frac{1}{3(1-p_1)}$	$2 - 3p_1$	1	$\frac{1}{2}$	$\frac{1}{2}$

Note that $0 \leq p_2 \leq 1$ implies $\frac{1}{3} \leq p_1 \leq \frac{2}{3}$, so the requirements for analogy-based expectations equilibria restrict the range of p_1 even without considering robustness.

10.2.1 Consistency Based Refinement

Expanding the consistency based refinement measures $\theta_1 \ , \ \theta_2$ when $N = 2$ gives:

$$\begin{aligned}
\theta_{1,l}(q \ , \ p) &= \frac{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})\,|\tilde{q}_l - q_n|}{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})} = \frac{|q_1 - \tilde{q}| + (1-q_1)(1-p_2)\,|q_2 - \tilde{q}|}{1 + (1-q_1)(1-p_2)} \\
\theta_{2,m}(p \ , \ q) &= \frac{\sum_{X_{n,1}\in\Psi_m} \Pr_{p,q}(X_{n,1})\,|\tilde{p} - p_n|}{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})} = \frac{|p_1 - \tilde{p}| + (1-p_1)(1-q_1)\,|p_2 - \tilde{p}|}{1 + (1-p_1)(1-q_1)}
\end{aligned}$$

This can be further simplified by recalling that consistency requires:

$$\begin{aligned}
\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2})\,(q_n - \tilde{q}) &= (1-p_1)(q_1 - \tilde{q}) + (1-p_1)(1-q_1)(1-p_2)(q_2 - \tilde{q}) = 0 \\
\sum_{X,1} \Pr_{p,q}(X_{n,1})\,(p_n - \tilde{p}) &= (p_1 - \tilde{p}) + (1-p_1)(1-q_1)(p_2 - \tilde{p}) = 0
\end{aligned}$$

[15] This was derived in Example 9.4 (see Table 9.2).

Therefore the consistency refinement measures can be written as:

$$\theta_1(q\ ,\ p) = \frac{2\,|q_1 - \bar{q}|}{1 + (1-q_1)\,(1-p_2)}$$

$$\theta_2(p\ ,\ q) = \frac{2\,|p_1 - \bar{p}|}{1 + (1-p_1)\,(1-q_1)}$$

Using Table 3.2 these can be expressed as a function of p_1:

$$\theta_1(q\ ,\ p) = \frac{|3p_1 - 1|}{2}$$

$$\theta_2(p\ ,\ q) = \frac{3}{4}\,|2p_1 - 1|$$

A judgement is needed to compare analogy classes between players, assuming they have similar sensitivities to the inconsistency between their opponent's real actions and the analogy expectations they hold. Therefore the robustness of an analogy-based expectations equilibrium depends on the least robust analogy class within that equilibrium. In other words the inconsistency of the equilibrium can be defined as

$$\theta^* = Max\,[\theta_{1,l}\ ,\ \theta_{2,m}]\ \forall\, l\ ,\ m$$

This is illustrated in Figure 10.5 which shows the mean absolute deviation of q, $\theta_1(q\ ,\ p)$, and p, $\theta_2(p\ ,\ q)$, resulting from the mixed strategy analogy-based expectations equilibrium given by p_1 when $N = 2$.

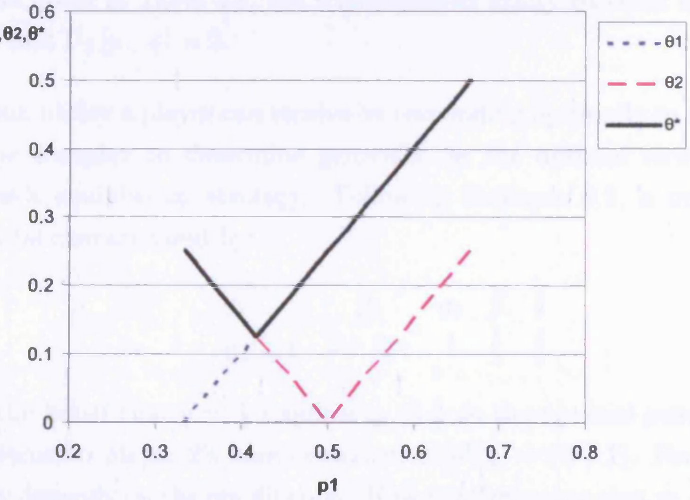


Figure 10.5: Consistency based refinement of mixed strategy equilibria when $N = 2$

As well as specifying how robust a given analogy-based expectations equilibrium is the question can be reversed to ask which analogy-based expectations equilibria survive

Therefore the consistency refinement measures can be written as:

$$\begin{aligned}\theta_1(q\ ,\ p) &= \frac{2\,|q_1 - \tilde{q}|}{1 + (1 - q_1)\,(1 - p_2)} \\ \theta_2(p\ ,\ q) &= \frac{2\,|p_1 - \tilde{p}|}{1 + (1 - p_1)\,(1 - q_1)}\end{aligned}$$

Using Table 3.2 these can be expressed as a function of p_1:

$$\begin{aligned}\theta_1(q\ ,\ p) &= \frac{|3p_1 - 1|}{2} \\ \theta_2(p\ ,\ q) &= \frac{3}{4}\,|2p_1 - 1|\end{aligned}$$

A judgement is needed to compare analogy classes between players, assuming they have similar sensitivities to the inconsistency between their opponent's real actions and the analogy expectations they hold. Therefore the robustness of an analogy-based expectations equilibrium depends on the least robust analogy class within that equilibrium. In other words the inconsistency of the equilibrium can be defined as

$$\theta^* = Max\,[\theta_{1,l}\ ,\ \theta_{2,m}]\ \forall\, l\ ,\ m$$

This is illustrated in Figure 10.5 which shows the mean absolute deviation of q, $\theta_1(q\ ,\ p)$, and p, $\theta_2(p\ ,\ q)$, resulting from the mixed strategy analogy-based expectations equilibrium given by p_1 when $N = 2$.

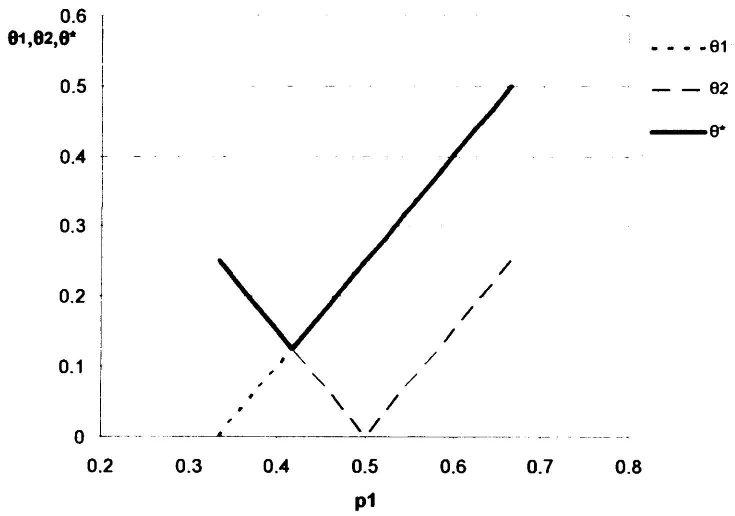


Figure 10.5: Consistency based refinement of mixed strategy equilibria when $N = 2$

As well as specifying how robust a given analogy-based expectations equilibrium is the question can be reversed to ask which analogy-based expectations equilibria survive refinement for a given robustness measure. A logical step following this is to investigate which analogy-based expectations equilibrium is the most robust given this method of refinement. This is illustrated in Figure 10.5 as the intersection between θ_1 and θ_2, i.e. the lowest point on θ^*. At this point $\theta_1 = \theta_2$ and so $\frac{|3p_1-1|}{2} = \frac{3}{4}|2p_1 - 1|$. This has a unique solution $p_1 = \frac{5}{12}$ giving $\theta_1 = \theta_2 = \frac{1}{8}$. Therefore the most robust equilibrium, where both players' analogy classes have the lowest possible mean absolute deviation, is that in which:

p_1	q_1	p_2	q_2	θ^*
0.4167	0.4286	0.75	1	0.125

Although each analogy class contains just two nodes, this refinement gives an equilibrium in which the observed deviation of beliefs is relatively small, especially compared to the pure strategy equilibria in Section 10.1. Secondly, pure *take* by player 2 is observed only with probability $\frac{1}{12}$ in this equilibrium.

10.2.2 Optimality Based Refinement

The robustness of mixed strategy equilibria in the Centipede and Doubling Dollar games illustrated in Figures 10.3 and 10.4 respectively will now be examined using optimality based refinement measures. Appendices B.1 and B.2 derive these measures fully for the Centipede and Doubling Dollar games respectively. A convenient feature of the Centipede game is that in all the mixed strategy analogy based expectations equilibria given in Table 3.2, the true expected utility received by each player is $U_1[p\ ,\ q] = 1$ and $U_2[p\ ,\ q] = 2$.

The maximum utility a player can receive by responding optimally to an opponent's strategy may be complex to determine generally, as the optimal strategy depends on the opponent's equilibrium strategy. Following Example 9.2, a mixed strategy equilibrium can be characterised by:

p_1	q_1	p_2	q_2	$\widetilde{p}$	$\widetilde{q}$
p_1	$1-\frac{1}{3(1-p_1)}$	$2-3p_1$	1	$\frac{1}{2}$	$\frac{1}{2}$

As $q_2 = 1$, the belief that $\widetilde{q} = \frac{1}{2}$ requires $q_1 \leq \frac{1}{2}$ so the optimal pure strategy for player 1 in response to player 2's true behaviour is $p^*(q) = (0\ ,\ 1)$. For player 2 the optimal strategy depends on the equilibrium. If $p_2 \leq \frac{1}{2}$ (meaning that $p_1 \geq \frac{1}{2}$) the best response is $q^*(p) = (0\ ,\ 1)$ while if $p_2 \geq \frac{1}{2}$ (meaning that $p_1 \leq \frac{1}{2}$) the best response is $q^*(p) = (1\ ,\ 1)$. The expected payoffs from these strategies are derived in Appendix B.1:

$$\begin{aligned} t_1(p\,,\,q) &= 1-2q_1 \\ t_2(p\,,\,q) &= \begin{cases} 1-2p_1 & \text{if } p_1 \leq \frac{1}{2} \\ 2-3p_1+2p_2(p_1-1) & \text{if } p_1 \geq \frac{1}{2} \end{cases} \end{aligned}$$

Substituting for the consistency and optimality conditions allows these to be expressed in terms of p_1:

$$\begin{aligned} t_1(p\,,\,q) &= \frac{2}{3\,(1-p_1)}-1 \\ t_2(p\,,\,q) &= \begin{cases} 1-2p_1 & \text{if } p_1 \leq \frac{1}{2} \\ 7p_1-6p_1^2-2 & \text{if } p_1 \geq \frac{1}{2} \end{cases} \end{aligned}$$

This is illustrated in Figure 10.6 which shows the (normalised) relative utility gains, $t_i(p\,,\,q)$, which player i makes if he switched to the true optimal strategy.

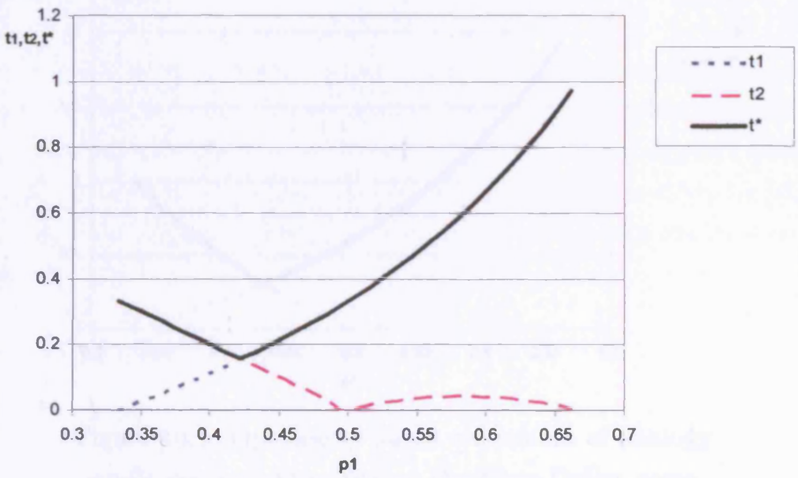


Figure 10.6: Optimality based refinement of analogy equilibria in the two period Centipede game

As under consistency based refinement, the most robust equilibrium under optimality based refinement is illustrated in Figure 10.6 as the intersection between t_1 and t_2, i.e. the lowest point on t^*. At this point $t_1 = t_2$ and so $1-2q_1 = 1-2p_1$, so $p_1 = q_1$. The behavioural strategies in this equilibrium are summarised as follows:

p_1	q_1	p_2	q_2	t^*
0.4226	0.4226	0.7321	1	0.1547

Note that $t^* = 0.1547$, so if either player were to switch to the true optimal strategy he would only increase his (normalised) payoff by 15.47%, despite each analogy class containing just two nodes. These equilibrium strategies for the Centipede game are close to those that are most robust to consistency based refinement Section 10.2.1. Pure *take* by player 2 is observed only with probability 0.0893 in this equilibrium.

Appendix B.2 analyses the optimality based refinement of the Doubling Dollar game, giving the following tolerance measures:

$$t_1(p\,,\,q) \;=\; \frac{2}{(4-3p_1)(1-p_1)} - 1$$
$$t_2(p\,,\,q) \;=\; \begin{cases} \frac{1-p_1}{p_1} - 1 & \text{if } p_1 \leq \frac{1}{2} \\ \frac{2(1-p_1)(3p_1-1)}{p_1} - 1 & \text{if } p_1 \geq \frac{1}{2} \end{cases}$$

This is illustrated in Figure 10.7, showing the (normalised) relative utility gains, $t_i(p\,,\,q)$, which player i could make by switching to the true optimal strategy.

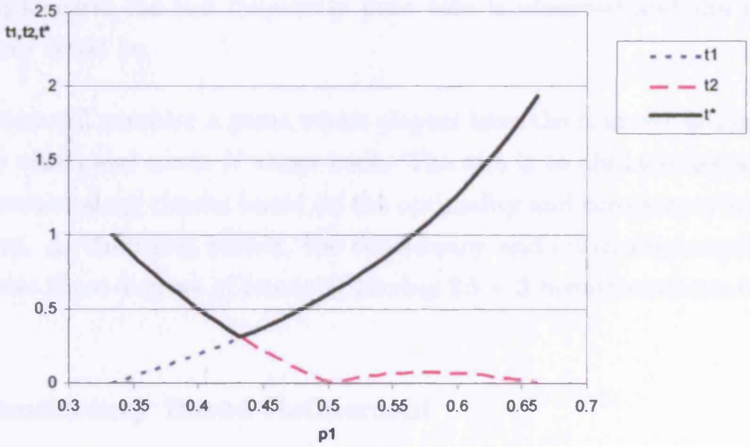


Figure 10.7: Optimality based refinement of analogy equilibria in the two period Doubling Dollar game

As in the Centipede game illustrated in Figure 10.6, the most robust equilibrium under optimality based refinement is illustrated in Figure 10.7 as the intersection between t_1 and t_2, i.e. the lowest point on t^*. At this point $t_1 = t_2$ and p_1 can be calculated by solving the equation $\frac{1-p_1}{p_1} - 1 = \frac{2(1-p_1)(3p_1-1)}{p_1} - 1$. The behavioural strategies in this equilibrium are summarised as follows:

p_1	q_1	p_2	q_2	t^*
0.4334	0.4117	0.6998	1	0.3074

As $t^* = 0.3074$, if either player were to switch to the true optimal strategy he increases his (normalised) payoff by 30.74%. While this is lower than the equivalent measure for the pure strategy equilibrium, partly because pure *take* is observed only with probability 0.1, it is still much higher than the Centipede optimality based refinement measure due to the greater impact of *taking* on the payoffs.

10.3 Refinement of Mixed Strategy Analogy Equilibria in N Period Games

The solutions in Section 10.2 assume $N = 2$. Given that the analogy classes contain only two nodes, some mixed strategy equilibria are surprisingly robust. For example, in the most robust mixed strategy analogy-based expectations equilibrium of the Centipede game, switching to the real optimal strategy increases (normalised) utility by just 15.47% for either player. One of the reasons mixed strategy analogy classes are expected to be more robust is that pure *take*, which must occur at the end of the game, is observed with low probability. Following this logic, the more nodes over which mixing occurs, the less frequently pure *take* is observed and the more robust analogy classes could be.

This section will consider a game where players have the coarsest analogy partition (one analogy class) and move N times each. The aim is to characterise and compare the most robust analogy classes based on the optimality and consistency based robustness measures. As discussed earlier, the consistency and optimality requirements for equilibrium use three degrees of freedom, leaving $2N - 3$ behavioural strategies as free parameters.

10.3.1 Consistency Based Refinement

This section is motivated by considering the most robust analogy class possible, when analogy beliefs and real actions correspond exactly so that $p = (\widehat{p},\ \widehat{p},\ ...,\ \widehat{p})$ and $q = (\widehat{q},\ \widehat{q},\ ...,\ \widehat{q})$. Unfortunately this is not an analogy-based expectations equilibrium, as player 2's optimal response in the final decision node, q_N, is to *take*. Increasing q_N to 1 satisfies optimality, but the probability of taking in some other nodes must be reduced below $\widehat{q}$ to maintain $\widetilde{q} = \widehat{q}$ which is necessary for player 1 to mix in equilibrium. The linearity of the problem suggests a bang-bang solution, regaining consistency by reducing the probability of taking, q_g, in a single node specified as $X_{g,2}$.[16]

Proposition 10.5 *The consistent strategy for player* 2 *which minimises absolute deviation is to mix with probability* $\widehat{q}$ *for all nodes except* $X_{1,2}$ *and* $X_{N,2}$.

[16]This section assumes that N is sufficiently large that the proposed analogy equilibrium exists. In the Centipede and Doubling Dollar games in Figures 2.2 and 2.3 it is only necessary that $N \geq 2$.

Proof. Assume that $p_n < 1 \; \forall \; n$ and player 2 adopts a strategy $q = (\widehat{q}, \; \widehat{q}, \; ..., \; 1)$. Player 1's analogy beliefs must be consistent so $\sum_{X_{n,2}} \Pr_{p,q} (X_{n,2}) (q_n - \widetilde{q}) = 0$. If player 2's behaviour at a single node $X_{g,2}$ is adjusted to give consistency, then $q_n = \widehat{q} = \widetilde{q} \; \forall \; n \neq g$, N. Therefore the consistency requirement becomes

$$\Pr_{p,q} (X_{N,2}) (1 - \widehat{q}) = \Pr_{p,q} (X_{g,2}) (\widehat{q} - q_g)$$

and therefore $\widehat{q} - q_g = (1 - q_g) (1 - \widehat{q})^{N-g} \prod_{n=g+1}^{N} (1 - p_n)$, so

$$q_g = 1 - \frac{1 - \widehat{q}}{1 - (1 - \widehat{q})^{N-g} \prod_{n=g+1}^{N} (1 - p_n)}$$

meaning that q_g is decreasing in g. The expression for absolute deviation is

$$\sum_{X_{n,2}} \Pr_{p,q} (X_{n,2}) |\widehat{q} - q_n| = 2 \Pr_{p,q} (X_{N,2}) (1 - \widehat{q}) = 2 \prod_{n=1}^{N} (1 - q_n) (1 - p_n)$$

which is decreasing in q_g, and as q_g is decreasing in g it follows that to minimise absolute deviation g should be set equal to 1. It is never optimal to set some $q_n > \widehat{q}$ as that means q_g will have to be reduced more to offset it, increasing the absolute deviation. The linearity of the absolute deviation in q_g means that this is a bang-bang solution, so $q_n = \widehat{q} \; \forall \; n \neq g$, N is in fact optimal. ∎

The intuition behind Proposition 10.5 is that to offset the final *take*, in some node $X_{g,2}$ player 2 must *take* with probability $q_g < \widehat{q}$. The earlier this occurs (i.e. the smaller g is), the more often it is observed and the more heavily it is weighted in expectations, so a smaller deviation from $\widehat{q}$ is sufficient to offset the final *take*. Therefore q_g is decreasing in g, so setting $g = 1$ maximises q_g. The direct effect of changing g on the absolute deviation is 0, because the although the deviation $|\widetilde{q} - q_g|$ increases as g increases, this is exactly offset by the fact that it is observed less frequently. There is a powerful indirect effect, however. The higher q_g is, the less frequently the final pure *take* is observed.

Proposition 10.6 *If $p_n = \overline{p} \; \forall \; n$ the strategy for player 2 which is consistent with $\widehat{q} = \widetilde{q}$ and maximises the expected number of nodes $X_{n,2}$ observed is to mix with probability $\widehat{q}$ for all nodes except $X_{1,2}$ and $X_{N,2}$.*

Proof. Assume that $p_n = \overline{p} \; \forall \; n$ and player 2 adopts a strategy $q = (\widehat{q}, \; \widehat{q}, \; ..., \; 1)$. Player 1's analogy beliefs must be consistent so $\sum_{X_{n,2}} \Pr_{p,q} (X_{n,2}) (q_n - \widetilde{q}) = 0$. If player 2's behaviour at a single node $X_{g,2}$ is adjusted to guarantee consistency, then from Proposition 10.5 $q_g = 1 - \frac{1-\widehat{q}}{1-(1-\widehat{q})^{N-g}\prod_{n=g+1}^{N}(1-\overline{p})}$. Substituting this into the expected number of times player 2 moves gives Equation 10.1.[17]

$$\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2}) = \frac{1-\overline{p}+\overline{p}\left[(1-\widehat{q})(1-\overline{p})\right]^{g}}{\widehat{q}+\overline{p}-\widehat{q}\overline{p}} \tag{10.1}$$

Equation 10.1 shows that $\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2})$ is decreasing in g, so minimising g maximises the number of times player 2 is expected to move in equilibrium. ■

When $p_n = \overline{p}\ \forall\ n$ and $q_g = 1 - \frac{1-\widehat{q}}{1-(1-\widehat{q})^{N-g}\prod_{n=g+1}^{N}(1-\overline{p})}$ the expected number of nodes observed in Equation 10.1 is independent of N. For example, if player 1 always *passes*, then $\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2}) = \frac{1}{\widehat{q}}$.

Proposition 10.7 *There is an analogy-based expectations equilibrium in which* $p_n = \widehat{p}\ \forall\ n$, $q_1 = 1 - \frac{1-\widehat{q}}{1-(1-\widehat{q})^{N-g}(1-\widehat{p})^{N-g}}$, $q_n = \widehat{q}\ \forall\ (1 < n < N)$, $q_N = 1$. *This is the most robust analogy-based expectations equilibrium to consistency based refinement given player 2's analogy class remains totally robust.*

Proof. If $p_n = \widehat{p}\ \forall\ n$ then $\overline{p} = \widehat{p}$, and $\overline{q} = \widehat{q}$ from the definition of q_1 in Proposition 10.5, therefore as both players have consistent beliefs and are indifferent at all nodes the proposed strategy is an analogy-based expectations equilibrium. The consistency requirement $\theta_{1,l}(q\ ,\ p) = \frac{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})|\overline{q}_l - q_n|}{\sum_{X_{n,2}\in\Omega_l} \Pr_{p,q}(X_{n,2})}$ is minimised by setting $g = 1$ as following Proposition 10.1, minimising g minimises the absolute deviation (the numerator) and following Proposition 10.2, minimising g maximises the expected number of nodes player 2 moves at (the denominator). Therefore minimising g also minimises $\theta_{1,l}(q\ ,\ p)$. For player 2, $\theta_{2,m}(q\ ,\ p) = 0$. ■

In fact, Proposition 10.7 follows from the linearity of the mean absolute deviation and holds very generally. It can be interpreted intuitively as a statement that the most robust consistent strategy for player 2 is to *take* with probability $q_1 < \widehat{q}$ in the first turn, mix with probability $\widehat{q}$ at all other turns and *take* in the last turn. As discussed in Chapter Nine, $\theta_{1,l}(q\ ,\ p)$ could be reduced further by allowing $\theta_{2,m}(q\ ,\ p) > 0$. To avoid repetition, an equivalent expression to Proposition 10.7 is not derived for player 1, but the intuition is the same. $\theta_{1,l}(q\ ,\ p)$ can be reduced by increasing the probability that player 1 *takes* in any node before it. The most robust way to achieve this is for

[17]This is calculated in Appendix B4.

player 1 to *take* with probability $p_N > \widehat{p}$, just before q_N is reached, and offset this by setting $p_1 < \widehat{p}$ while leaving $p_n = \widehat{p}\ \forall\ (1 < n < N)$. Conjecture 10.1 follows directly from these intuitions and will be illustrated with examples. Assuming a degree of symmetry between players, the minimum robustness will set $\theta_{1,l}(q\ ,\ p) = \theta_{2,m}(q\ ,\ p)$.

Conjecture 10.1 *In the analogy-based expectations equilibrium most robust to consistency based refinement, players use the strategies illustrated in Table* 10.2.

q_1	q_1	$\cdots$	q_n	$\cdots$	q_N
$1-\frac{1-\widehat{q}}{1-(1-\widehat{q})^{N-1}(1-\widehat{p})^{N-2}(1-p_N)}$	$\widehat{q}$		$\widehat{q}$		1

p_1	p_1	$\cdots$	p_n	$\cdots$	p_N
$1-\frac{1-\widehat{p}}{1-(1-q_1)(1-\widehat{q})^{N-2}(1-\widehat{p})^{N-2}(p_N-\widehat{p})}$	$\widehat{p}$		$\widehat{p}$		$\approx\frac{1+\widehat{p}}{2}$

Table 10.2. *Most robust analogy class to consistency based refinement*

Generally p_N *is set so that* $\theta_{1,l}(q\ ,\ p) = \theta_{2,m}(q\ ,\ p)$. *When* $\widehat{q} = \widehat{p}$, $p_N \approx \frac{1+\widehat{p}}{2}$

Therefore in the Centipede and Doubling Dollar games illustrated in Figures 10.3 and 10.4, where $\widehat{q} = \widehat{p} = \frac{1}{2}$ and $N = 2$, the most robust analogy-based expectations equilibrium is that in which $q_2 = 1$, $p_2 \approx \frac{1+\widehat{p}}{2} = \frac{3}{4}$, $q_1 = \frac{3}{7}$, $p_1 = \frac{5}{12}$. These exactly equal the behavioural strategies derived above and illustrated in Figure 10.5. Example 10.1 illustrates Proposition 10.7, the equilibrium most robust to consistency refinement when $p = (\widehat{p},\ \widehat{p},\ \widehat{p},\ \widehat{p})$. Example 10.2 shows that the corresponding solution when $g = 3$ is inferior. Example 10.3 demonstrates that increasing $\theta_{2,m}$ (by changing p) can further reduce $\theta_{1,l}(q\ ,\ p)$.

Example 10.1 *Consider the Centipede and Doubling Dollar games with* $N = 4$ *illustrated in Figures* 8.2 *and* 8.3. *Following Proposition* 10.7, *an analogy-based expectations equilibrium is* $p = (\widehat{p},\ \widehat{p},\ \widehat{p},\ \widehat{p}) = (0.5,\ 0.5,\ 0.5,\ 0.5)$ *and* $q = \left(\left[1-\frac{1-\widehat{q}}{1-(1-\widehat{q})^{N-1}(1-\widehat{p})^{N-1}}\right],\ \widehat{q},\ \widehat{q},\ 1\right) = (0.492,\ 0.5,\ 0.5,\ 1)$. $\theta_{1,l}(q\ ,\ p) = 0.0119$ *and* $\theta_{2,m}(q\ ,\ p) = 0$. *This equilibrium is the most robust to consistency based refinement without adjusting player* 1*'s strategy.*

Example 10.2 *Consider the Centipede and Doubling Dollar games with* $N = 4$ *illustrated in Figures* 8.2 *and* 8.3. *An analogy-based expectations equilibrium is* $p = (\lesssim \widehat{p},\ \widehat{p},\ \widehat{p},\ > \widehat{p}) = (0.5,\ 0.5,\ 0.5,\ 0.5)$ *and* $q = (\lesssim \widehat{q},\ \widehat{q},\ \widehat{q},\ 1) = (0.5,\ 0.5,\ 0.3333,\ 1)$. $\theta_{1,l}(q\ ,\ p) = 0.0156$ *and* $\theta_{2,m}(q\ ,\ p) = 0$. *Therefore setting* $g = 3$ *significantly increases* $\theta_{1,l}(q\ ,\ p)$, *reducing robustness, compared to the case when* $g = 1$ *in Example* 10.1.

Example 10.3 *Consider the Centipede and Doubling Dollar games with* $N = 4$ *illustrated in Figures* 8.2 *and* 8.3. *Following Conjecture* 10.1, *an analogy-based expectations* *equilibrium is* $p = (\lesssim \widehat{p},\ \widehat{p},\ \widehat{p},\ > \widehat{p}) = (0.4960,\ 0.5,\ 0.5,\ 0.75)$ *and* $q = (\lesssim \widehat{q},\ \widehat{q},\ \widehat{q},\ 1) = (0.4961,\ 0.5,\ 0.5,\ 1)$. $\theta_{1,l}(q\ ,\ p) = \theta_{2,m}(q\ ,\ p) = 0.0060$. *This equilibrium is the most robust to consistency based refinement.*

Example 10.3 shows that even when $N = 4$, a mixed strategy analogy-based expectations equilibrium can be extremely robust to consistency based refinement, with a mean absolute deviation as low as 0.0060. The final *take* is observed with a probability less than $\frac{1}{250}$, so it is very unlikely players will realise behaviour at node $X_{N,2}$ is significantly different from $\widehat{q}$. Of course, players need not coordinate on the most robust mixed strategy analogy-based expectations equilibria, though any equilibrium in this region is extremely robust to consistency based refinement. The thesis will now derive a very similar result under optimality based refinement.

10.3.2 Optimality Based Refinement

As with consistency based refinement, this solution is motivated by considering the most robust analogy class possible, when analogy beliefs and real actions correspond so $p = (\widehat{p},\ \widehat{p},\ ...,\ \widehat{p})$ and $q = (\widehat{q},\ \widehat{q},\ ...,\ \widehat{q})$. In this case players are truly indifferent at all nodes, so mixed strategies are optimal. This is not an analogy-based expectations equilibrium, as optimally, $q_N = 1$. In this case, however, the probability of player 2 taking in some other nodes must be reduced below $\widehat{q}$ to maintain player 1's analogy belief $\widetilde{q} = \widehat{q}$. The linearity of the problem leads to a bang-bang solution, regaining consistency by reducing the probability of taking q_g in just one node $X_{g,2}$.[18]

Proposition 10.8 *The consistent strategy for player* 2 *which minimises optimal utility is to mix with probability* $\widehat{q}$ *for all nodes except* $X_{1,2}$ *and* $X_{N,2}$.

Proof. Assume that $p_n < 1\ \forall\ n$ and player 2 adopts a strategy $q = (\widehat{q},\ \widehat{q},\ ...,\ 1)$. Player 1's analogy beliefs must be consistent so $\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2})(q_n - \widetilde{q}) = 0$. If player 2's behaviour at a single node $X_{g,2}$ is adjusted to guarantee consistency, then $q_n = \widehat{q} = \widetilde{q}\ \forall\ n \neq g,\ N$. Therefore the consistency requirement becomes:

$$\Pr_{p,q}(X_{N,2})(1-\widehat{q}) = \Pr_{p,q}(X_{g,2})(\widehat{q} - q_g)$$

and can be rewritten $\widetilde{q} - q_g = (1-q_g)(1-\widehat{q})^{N-g} \prod_{n=g+1}^{N}(1-p_n)$ therefore $q_g = 1 - \frac{1-\widehat{q}}{1-(1-\widehat{q})^{N-g}\prod_{n=g+1}^{N}(1-p_n)}$ so q_g is decreasing in g. For player 1, as player 2 mixes with

[18] This section assumes that the game is sufficiently long that the analogy equilibrium proposed in the following sections exist. In the Centipede and Doubling Dollar games in Figures 8.2 and 8.3 it is only necessary that $N \geq 2$.

probability $\widehat{q}$ at all nodes after q_g, a weakly optimal strategy is to always *pass* and *take* in $X_{N,1}$.[19] As the payoff in node $X_{N,1}$ is $\frac{\mu(1-\lambda^N)}{1-\lambda} + \lambda^N$, the expected payoff from this strategy is $\left[\frac{\mu(1-\lambda^N)}{1-\lambda} + \lambda^{N-1}\right](1-\widehat{q})^{N-2}(1-q_g)$, which is decreasing in q_g. Therefore as q_g is decreasing in g it follows that to minimise optimal utility g should be set equal to 1. It is never optimal to set some $q_n > \widehat{q}$ as that means q_g would have to be reduced more to offset it, increasing the absolute deviation. The linearity of the absolute deviation in q_g means that this is a bang-bang solution, so $q_n = \widehat{q} \, \forall \, n \neq g, N$ is in fact optimal. ■

Unfortunately, the true expected true utility is increasing in g. However, even when mixing is over just 2 nodes the deviations of q_1 and p_1 are sufficiently small that the true expected utility is almost constant for different values of g. In fact, when q_g and p_g are set to provide consistency in the Centipede game illustrated in Figure 8.2, for all g, $U_1[p\ ,\ q] = 1$ and $U_2[p\ ,\ q] = 2$ and so expected true utility is constant. In the Doubling Dollar game illustrated in Figure 8.3, then for all g, $U_1[p\ ,\ q] \approx 0.97$ and $U_2[p\ ,\ q] \approx 0.48$. On the other hand, the effect of changing g on optimal utility, $U_1[p^*(q)\ ,\ q]$, is significant, so it is the effect derived in Proposition 10.8 that determines $t_1(p\ ,\ q)$ and $t_2(p\ ,\ q)$. This is illustrated by Examples 10.4 and 10.5.

Example 10.4 *Consider the Doubling Dollar game with* $N = 4$ *illustrated in Figure* 8.3. *An analogy-based expectations equilibrium of this game is given by* $p = (\widehat{p}, \widehat{p}, \widehat{p}, \widehat{p}) = (0.5, 0.5, 0.5, 0.5)$ *and* $q = (\lesssim \widehat{q}, \widehat{q}, \widehat{q}, 1) = (0.492, 0.5, 0.5, 1)$. $t_1(p\ ,\ q) = 0.076$ *and* $t_2(p\ ,\ q) = 0$. *This is the most robust analogy class to optimality based refinement without increasing* $t_2(m)(p\ ,\ q)$ *by changing* p.

Example 10.5 *Consider the Doubling Dollar game with* $N = 4$ *illustrated in Figure* 8.3. *An analogy-based expectations equilibrium of this game is given by* $p = (\lesssim \widehat{p}, \widehat{p}, \widehat{p}, > \widehat{p}) = (0.5, 0.5, 0.5, 0.5)$ *and* $q = (\lesssim \widehat{q}, \widehat{q}, \widehat{q}, 1) = (0.5, 0.5, 0.333, 1)$. $t_1(p\ ,\ q) = 0.391$ *and* $t_2(p\ ,\ q) = 0$. *This is significantly less robust than the analogy equilibrium in Example* 10.4 *in which* $g = 1$.

In Example 10.4, consistency is achieved by player 2 offsetting the *take* at $X_{N,2}$ by passing slightly more often in the first turn (i.e. $g = 1$). This analogy-based expectations equilibrium is the most robust possible in the Doubling Dollar game when $N = 4$, given player 1's strategy. Player 1 could increase his payoffs by 7.6% by reacting optimally to their opponent's true behaviour, and taking in node $X_{N,1}$.

[19] Any strategy in which player 1 *passes* until $X_{g,2}$ and *takes* with probability 1 before $X_{N,2}$ is weakly optimal.

In example 10.5 however, consistency is achieved by player 2 offsetting the *take* at $X_{N,2}$ by passing more often in node $X_{3,2}$ (i.e. $g = 3$). In this case the analogy-based expectations equilibrium is significantly less robust, as player 1 could increase his utility by 39.1% if he adjusted his strategy to be an optimal response to q.

Example 10.6 *Consider the Centipede game with $N = 4$ illustrated in Figure 8.2. An analogy-based expectations equilibrium of this game is given by $p = (\lesssim \widehat{p},\ \widehat{p},\ \widehat{p},\ > \widehat{p}) = (0.4960,\ 0.5,\ 0.5,\ 0.7490)$ and $q = (\lesssim \widehat{q},\ \widehat{q},\ \widehat{q},\ 1) = (0.4960,\ 0.5,\ 0.5,\ 1)$. $t_1(p\ ,\ q) = t_2(p\ ,\ q) = 0.0079$. This is the most robust analogy class to optimality based refinement.*

Example 10.7 *Consider the Doubling Dollar game with $N = 4$ illustrated in Figure 8.3. An analogy-based expectations equilibrium of this game is given by $p = (\lesssim \widehat{p},\ \widehat{p},\ \widehat{p},\ > \widehat{p}) = (0.4972,\ 0.5,\ 0.5,\ 0.6793)$ and $q = (\lesssim \widehat{q},\ \widehat{q},\ \widehat{q},\ 1) = (0.4949,\ 0.5,\ 0.5,\ 1)$. $t_1(p\ ,\ q) = t_2(p\ ,\ q) = 0.0474$. This is the most robust analogy class to optimality based refinement.*

Example 10.6 shows that most robust analogy-based expectations equilibrium of the Centipede game is significantly more robust than that of the Doubling Dollar game in Example 10.7. Consistency is achieved by player 2 offsetting the *take* at $X_{N,2}$ by passing slightly more often in the first turn, $X_{1,2}$ (i.e. $g = 1$) The condition for requirement is balanced across the two players as player 1 *takes* slightly more in node $X_{N,1}$ and maintains consistency by passing slightly more in the first turn. In the equilibrium of the Doubling Dollar game in Example 10.7, players could increase their (normalised) utilities by 4.74% by responding optimally to their opponent's true behaviour. In the Centipede game, switching to the true optimal response leads to gains of only 0.79%.

Conjecture 10.2 *The analogy-based expectations equilibrium most robust to optimality based refinement sets:*

q_1	q_1	$\cdots$	q_n	$\cdots$	q_N
$1 - \frac{1-\widehat{q}}{1-(1-\widehat{q})^{N-1}(1-\widehat{p})^{N-2}(1-p_N)}$	$\widehat{q}$		$\widehat{q}$		1

p_1	p_1	$\cdots$	p_n	$\cdots$	p_N
$1 - \frac{1-\widehat{p}}{1-(1-q_1)(1-\widehat{q})^{N-2}(1-\widehat{p})^{N-2}(p_N-\widehat{p})}$	$\widehat{p}$		$\widehat{p}$		$\gtrsim \widehat{p}$

Table 10.3. Most robust analogy class to optimality based refinement

and p_N is set so that $t_1(p\ ,\ q) = t_2(p\ ,\ q)$

Conjecture 10.2 follows from the discussion and examples in this section. Overall, the equilibrium which is most robust to optimality based refinement is very similar to that which is most robust to optimality based refinements, supporting the intuition developed in the case when $N = 2$.

10.4 Discussion

This section will summarise some of the main results from endogenising analogy classes, before extending the analysis to the case of multiple analogy classes in Chapter Eleven.

Different Payoff Structures

When refinement is based on the variation of behaviour within an analogy class, the payoff structure of the game is irrelevant for a given $\widehat{p}$ and $\widehat{q}$. Therefore any consistency based refinement of the Centipede game in Figure 8.2 applies equally to the Doubling Dollar game illustrated in Figure 8.3. This is *not* true of optimality based refinement. Proposition 10.4 showed that the robustness of the pure strategy equilibrium involving passing is increasing in μ and decreasing in λ. The pure strategy passing equilibrium in the Doubling Dollar game, in which player 2 *takes* in the final node, is never robust to optimality based refinement, no matter how long the game. However, there are mixed strategy analogy-based expectations equilibria which may be very robust if the game is sufficiently long. For a given game length and equilibrium,[20] the Centipede game is more robust than the Doubling Dollar game. More generally, for a given $\widehat{p}$ and $\widehat{q}$,[21] the lower the value of λ the more robust a mixed strategy analogy-based expectations equilibrium is to optimality based refinement. When λ and μ are increased individually however, without keeping $\lambda + \mu$ constant, then over a large domain of parameter values an indirect effect dominates these direct effects: increasing λ or μ individually raises $\widehat{p}$ and $\widehat{q}$. In robust analogy-based expectations equilibrium this can reduce the probability with which the pure *take* node is observed and also brings this pure *take* closer to $\widehat{q}$ (or $\widehat{p}$) the expected behaviour necessary for indifference.

Passing and Mixing Equilibria

Mixed strategy analogy-based expectations equilibria may survive refinement in games which do not have a robust pure strategy equilibrium involving passing. For example, the only pure strategy equilibrium that is robust to optimality based refinement in the Doubling Dollar game has the subgame-perfect outcome. However, robust mixed strategy equilibria may exist even for relatively small values of N, as shown in Example 10.7 where $t_1(p\ ,\ q) = t_2(p\ ,\ q) = 0.0474$. In the Centipede game the pure strategy passing equilibrium is robust for any level of player sensitivity, providing the game is long enough; Proposition 8.3 showed that when $N = 4$, $\theta_1 = 2\frac{N-1}{N^2} = 0.375$ and the

[20] Recall from Proposition 9.1 that a mixed strategy analogy-based expectations equilibrium of the centipede game is also a mixed strategy equilibrium of the doubling dollar game.

[21] If $\widehat{q} = \frac{\mu+\lambda-1}{\mu+\lambda}$ if the game is normalised, so maintaining $\widehat{q}$ means raising λ and simultaneously lowering μ.

optimality based refinement measure $t_1 = \frac{1}{N-1} = 0.333$. However, for a given length of the game, N, the mixed strategy equilibria are dramatically more robust than the equilibrium in pure strategies. The consistency and optimality based refinement measures for the most robust mixed strategy equilibrium, derived in Examples 10.2 and 10.7, were $\theta_1 = 0.0060$ and $t_1 = 0.0079$. To achieve the same degree of payoff robustness, the passing equilibrium requires a Centipede game with at least 126 moves for each player; the same degree of consistency based robustness needs $N = 333$.

Games of Different Lengths

It was demonstrated in Proposition 9.4 that the pure strategy analogy-based expectations equilibrium of the Centipede game involving passing is robust to refinement providing the game is long enough, although increasing N could not make the pure strategy passing equilibrium of the Doubling Dollar game robust to optimality based refinement. In mixed strategy equilibria, increasing N also reduces the probability of observing player 2 taking with probability 1 in node $X_{N,2}$. Therefore increasing N tends to increase robustness using either consistency or optimality based refinements when players mix. Using the intuition that the most robust analogy class involves passing slightly more often in turn $X_{1,2}$ to give robustness, approximations for consistency measures can be calculated for the Centipede and Doubling Dollar games for a given value of N. This is illustrated in Figure 10.8.

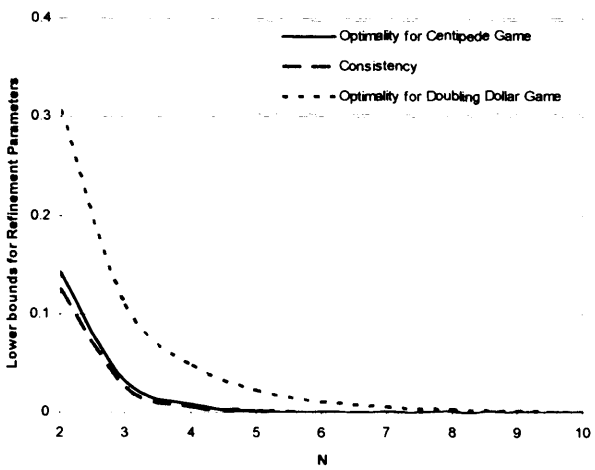


Figure 10.8: Lower bounds on Player's Refinement Parameters given N

From Figure 10.8 it is clear that a mixed strategy analogy-based expectations equilibrium when $N \geq 8$ can be extremely robust, even in the Doubling Dollar game.

Chapter 11

Extensions and Discussion

The purpose of this analysis has been to investigate which analogy-based expectations equilibria are robust when players form their analogy classes endogenously. An analogy class is less likely to be refined when it is a reasonable approximation of the opponent's true behaviour and when the utility gain from such refinement is small. Although the most robust equilibrium is characterised for illustrative purposes, refinement to a unique equilibria is not in the spirit of the analogy class approach. Rather, the aim is to provide bounds to rule out unrealistic behaviour for a given sophistication level of players. This section will discuss some straightforward extensions of the ideas presented before relating them to the literature on bounded rationality in finite horizon games in Chapter Twelve.

11.1 Multiple Analogy Classes

The ideas presented in Chapter Ten for refining single analogy classes extend to when players form multiple analogy classes. The robustness of an analogy-based expectations equilibrium depends on the least robust analogy class within the equilibrium, so it is important to refine *one analogy class at a time.* Firstly, this means assessing each analogy class conditional on it being reached. Secondly, the payoffs in the final node of an analogy class should be assessed using the analogy-based expectations of future behaviour.

If players have multiple analogy classes and play pure strategies, off-the-equilibrium-path beliefs are an issue if *take* occurs before the final node. However, this only arises when a player's best response to an analogy class is strictly to *take*, so in the mixed strategy equilibria developed in Chapter Nine, *take* can occur before the end of the game without this problem arising. For example, consider the following analogy-based expectations equilibrium of the four period Centipede game depicted in Figure 8.2. Players 1 and 2 use behavioural strategy profiles $p = \left(\frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, p_4\right)$ and $q = \left(\frac{3}{7},\, \frac{1}{2},\, 1,\, 1\right)$[1] respectively and group nodes in which their opponent moves in a single analogy class so that $\widetilde{q} = \widetilde{p} = \frac{1}{2}$. $q_4 = 1$ is required for optimality, and p_4 is irrelevant - both for the analogy-based expectations equilibrium and for the refinement measures proposed in this thesis. Taking at node $X_{2,3}$ is optimal for player 2 as beliefs off the equilibrium path are specified as $\widetilde{p} = \frac{1}{2}$ by the analogy class. If there is a cost of additional analogy classes this could remove classes to which the best response is *take*, if they follow a class in which mixing is expected, as the same strategy could be motivated by a single analogy class in which the opponent is expected to mix. More generally however, the analogy-based expectations equilibrium approach is consistent with players learning to use backwards induction and forming analogy classes finely in the last few nodes of the game.

Focusing on analogy classes in which players believe the opponent either *passes* or plays mixed strategies, the approach proposed in this thesis provides an intuitive solution to the finite horizon problem: an equilibrium consists of players passing for a given number of nodes and then mixing near the end of the game.[2] A top down view of the Centipede game when $N = 8$ is illustrated in Figure 11.1, in which at each node *take* is equivalent to playing down. In this example, each player forms two analogy classes. In Ω_1 player 1 expects player 2 to *pass*, while in Ω_2 player 1 expects player 2 to *take* with probability $\widehat{q}$ (which leaves player 1 indifferent between passing and taking). Player 2 expects player 1 to *pass* in Ψ_1 and mix with probability $\widehat{p}$ in Ψ_2.

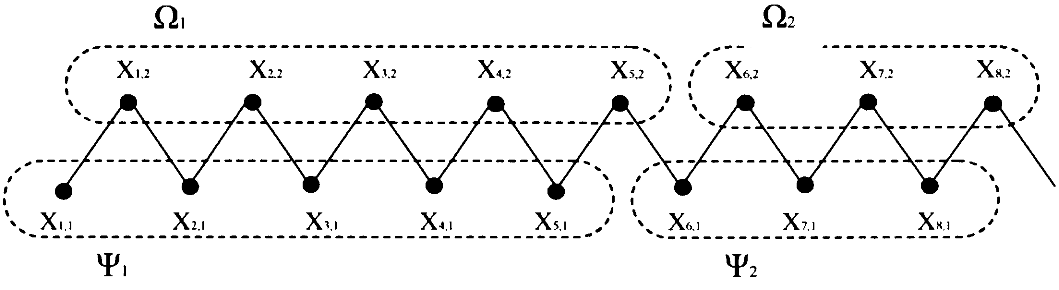


Figure 11.1: Top down view of the Centipede game when N=8

This approach might be even better motivated in longer finitely repeated games. For example, if $N = 100$ then the passing stage might last for 90 turns, followed by a 10 period mixing stage. As discussed, pure *take* could occur before the end of the game. The consistency and optimality based refinement measures, proposed in Chapter Ten,

[1] Finding a consistent value for q_1 follows by putting $k = 1$ and $N = 3$ in Proposition 4.7: $q_1 = 1 - \frac{1-\widehat{q}}{1-(1-\widehat{q})^2(1-\widehat{p})^2} = \frac{3}{7}$

[2] Player 2 must *take* with probability 1 in the final node.

mean that the sensitivity level of players puts a lower bound on the number of nodes in Ω_2 and Ψ_2, the analogy classes in which mixing is expected, for the equilibrium to be robust. However, as discussed in Section 10.4 and illustrated in Figure 10.8, an analogy class in which players expect the opponent to mix need only contain a few nodes to achieve this. The refinement measures do not put an upper bound on the number of turns of mixing required for an equilibrium to be robust, so for a given level of player sensitivity, there are multiple robust analogy equilibria that players might learn to coordinate on. The most cooperative of these is that in which pure *take* occurs only in node $X_{N,2}$, players use the most robust equilibrium behavioural strategies in the mixing stage of the game, and the mixing stage is as short as possible subject to achieving a given level of robustness.

When multiple analogy classes are formed, other issues include the structure of analogy classes, whether a passing stage could arise after a mixing stage and if mixing could occur when a player is expected to *pass*. Firstly, for a player to be indifferent between passing and taking it is necessary that he believes the opponent is mixing, so the analogy-based expectations equilibrium restricts the structure of analogy classes and behaviour even before refinement is applied. For example in Figure 11.1, if Ω_1 and Ω_2 were combined into a single analogy class Ω, then player 1 would expect player 2 to *pass* with high probability at all nodes. Player 1 would *pass* as a best response,[3] making player 2's expectation that player 1 mixes in Ψ_1 inconsistent, breaking the equilibrium. This argument means that the boundaries between player 1's analogy classes Ψ_1 and Ψ_2 and the boundary between player 2's analogy classes Ω_1 and Ω_2 must be close together, as shown in Figure 11.1.

Secondly, at node $X_{5,2}$ in Figure 11.1 player 2 is indifferent between passing and taking, yet player 1 expects him to *pass*. For player 2, passing in $X_{5,2}$ is among the best responses to his analogy beliefs, and in the most robust analogy class player 1 mixes with $p_5 < \widehat{p}$ at node $X_{6,1}$, so in this case it is generally optimal to pass in $X_{5,2}$. However, if player 2 mixed at node $X_{5,2}$, this is consistent with the approach, providing player 2 *takes* with relatively low probability, so player 1's analogy class, Ω_1, remains robust.

Finally, although the analogy-based expectations equilibrium does not rule out a phase in which players are expected to pass following a mixing phase, an equilibrium in which this is the case typically has very high consistency and optimality based refinement measures, and so would not be robust. For example, consider Figure 11.1

[3] If responding to Ω_2, player 1 is indifferent between *passing* and *taking*, then combining this with Ω_1 will leave player 1 with a best response of *pass*.

and suppose that players expect their opponents to mix in analogy classes Ω_1 and Ψ_1 but *pass* with high probability in analogy classes Ω_2 and Ψ_2. The expected passing in Ω_2 and Ψ_2 means players are no longer indifferent at all nodes in Ω_1 and Ψ_1. To mix, they must be indifferent only at nodes $X_{1,1}$ and $X_{1,2}$, in which they take with high probability so the opponent's expectation that they mix is consistent. For player 2 this strategy is very suboptimal as it involves taking with a high probability even though player 1 would actually *pass* for the next seven nodes! Although not considered explicitly, a similar argument can be applied to analogy classes that are not formed over an interval.[4]

In summary, it has been argued that an analogy class in which players expect their opponents to mix is generally necessary for robustness and that such a mixing phase is likely to be preceded by a period in which both players *pass*. The most cooperative robust analogy based expectations equilibrium is that in which pure *take* occurs only in node $X_{N,2}$, players use the most robust equilibrium behavioural strategies in the mixing stage of the game, and the mixing stage is as short as possible (subject to satisfying a given level of robustness).

11.2 Monotonicity

The probability of each player taking in an analogy-based expectations equilibrium need not be monotonically increasing. However, the previous section argued that when players have multiple analogy classes, an equilibrium in which passing behaviour is expected to follow mixing behaviour will not be robust. In the most robust mixed strategy equilibria, described in Conjectures 10.1 and 10.2, the probably of a player passing is weakly monotonic. This is because players offset taking with high probability at the end of the game by taking with $p < \widehat{p}$ in the first node of a mixed strategy analogy class. More generally, if an analogy class in which a player is expected to mix is reasonably long, behaviour towards the end of it is rarely observed, so non-monotonic strategies are possible. In experiments, monotonicity may be exaggerated due to heterogeneity between players. Taking the approach developed in this thesis, some players could coordinate on an analogy classes in which they expect one another to mix over the last 10 turns while others expect mixing in the last 9 turns. An experimental study would average these behaviours and observe a monotonically increasing probability of taking when in fact most mixing occurs with similar probabilities in these heterogeneous equilibria.

[4] In the sense that it would mean passing is expected after a period of mixing.

11.3 Different Games

This thesis has focused on games in which the problem faced by players is similar at every node, so behavioural strategies exist as described in Proposition 8.1, in which players are indifferent between passing and taking at every node. The analysis extends directly to the case when a game possesses this property only over some range (in which mixing occurs). By changing the way in which players form analogy expectations, however, the approach could be applied to a broader range of games. For example, rather than forming an analogy that "the opponent mixes with probability $\widetilde{p}$" players could instead form the analogy "the opponent mixes in such a way that I am indifferent at every node within Ω_2". This could still generate a specific value of $\widetilde{p}$ that is required for consistency, or it could be defined more broadly. This is related to Fudenberg and Levine's (1993) notion of a self-confirming equilibrium, in which players form consistent beliefs on the equilibrium path, and respond optimally to these beliefs, but adjustment does not take place on information sets that are not visited in equilibrium. However, if the period of learning were long enough this concept would still lead to a unique self-confirming equilibrium which is equivalent to the subgame-perfect equilibrium. More generally, analogies such as "the opponent mixes between taking and a trigger strategy at the end of the game" might permit robust cooperation in almost perfect information games such as the finitely repeated prisoner's dilemma.

11.4 Generalising the Solution Concept

In this thesis the focus has been on extending the analogy-based expectations equilibrium to the case when analogy classes are formed endogenously in finite horizon paradox games. One of the refinements was based on consistency, restricting the range of behaviour that players would group in an analogy. Although it is beyond the scope of this thesis, this approach could be endogenised as part of the requirement for an analogy equilibrium. It was argued that mixing is particularly robust to refinement in finite horizon problems. It is possible that being indifferent is considered focal, so players form the belief that the opponent is mixing with probability $\widehat{p}$ even though true behaviour may deviate from this slightly. Rather than requiring that an analogy-based expectation must equal a specific value for consistency, the requirement could be that the mean absolute deviation or variance of behaviour from $\widehat{p}$ must be less than some constant η. This approach could be considered similar to an ε equilibrium,[5] although in this case it is average beliefs which can deviate slightly from a focal analogy "the opponent mixes in a way which leaves me indifferent". If players form their beliefs

[5] ε- equilibrium (Radner, 1980) is discussed in more detail in Chapter Twelve.

from a finite history, consistency based on the variance of true behaviour around the focal belief could be similar to a statistical hypothesis test (weighted appropriately by frequency of observation). This approach introduces consistency based refinement into the notion of analogy-based expectations equilibrium directly, while maintaining full optimality of player's best responses. This addresses the idea that it is strategic uncertainty which leads to complexity in forming beliefs, in contrast to a difficulty in optimising when players form beliefs correctly. When $\eta = 0$ this approach leads to the unique subgame-perfect equilibrium, but mixed strategy analogy-based expectations equilibria may satisfy this measure of consistency even if η is very small.

Chapter 12

Relation to the literature

Chapter Eleven argued that the approach proposed in this thesis provides an intuitive solution to the finite horizon problem: an equilibrium consists of players passing for a given number of nodes and then mixing towards the end of the game. Such a solution was illustrated in Figure 11.1, which shows a top down view of the Centipede game, where at every node *take* is equivalent to playing down. In this example, each player forms two analogy classes. In Ω_1 and Ψ_1 players expect their opponent to *pass*, while in Ω_2 and Ψ_2 they expect their opponent to mix in a way that leaves them indifferent between mixing and passing. Chapter Ten showed that when players form their analogy classes endogenously, a mixed strategy equilibrium can be extremely robust to consistency and optimality based refinement. This chapter will relate the concepts underlying this approach to other attempts to resolve *finite horizon paradoxes*. Three particularly relevant approaches are those of machine games, incomplete information and models which argue for simplicity of beliefs. The discussion will finish by considering other important approaches.

Incomplete Information

Kreps *et al.* (1982) show that long periods of passing can be sustained in the finitely repeated prisoner's dilemma with incomplete information. Specifically, there is a small chance that one of the players is irrational and plays a tit-for-tat strategy. Even when the chance of being irrational is very small, it is optimal for rational players to maintain a reputation for being *tit-for-tat* players to convince the opponent to cooperate until late in the game. A similar approach can be used to generate passing in the Centipede game when there is a small probability that player 2 is a *cooperative type* who plays the strategy *always pass*. Players update their beliefs about whether the opponent is cooperative using Bayes' law (this is necessary even in the Centipede game as players mix at some point). A perfect Bayesian equilibrium can be sustained in which players *pass* for some periods, followed by mixing, then finally taking. Although this approach generates a similar equilibrium to that proposed in this thesis, the motivation is very different. In the mixed strategy analogy-based expectations equilibria, players are boundedly rational and form simple beliefs, for example that the opponent *passes* in the first 70 nodes of a centipede game and mixes in the last 30. In the incomplete information model of the Centipede game, players update their beliefs about the probability that their opponent is crazy by applying Bayes' law throughout the game, effectively making belief formation more complex.

Refinement of Beliefs

In the Centipede game, Spiegler (2002) demonstrates that *always pass* may be an equilibrium in justifiable strategies. This is a procedurally rational concept based on the idea that players need to justify their strategies *ex post* by offering a hypothesis about their opponent's strategy. The hypothetical strategies are simple, in the sense that they are modelled as single state finite automata.[1] As well as the subgame-perfect equilibrium, an equilibrium in justifiable strategies exists for the Centipede game in which player 1 always passes and player 2 takes only in the final period. Although these are the two pure strategy equilibria of the analogy-based expectations approach when players have the coarsest analogy groupings,[2] this can be contrasted with the importance placed on mixed strategies in this analysis.

Naturally the approach taken in this thesis closely follows Jehiel (2005), which introduces the analogy-based expectations equilibrium and applies it to a broader range of games. The contribution of this thesis is to endogenise the formation of analogy classes within a specific class of games, and use this to motivate the importance of mixed strategies when forming analogies in finite horizon problems. Another approach is taken by McKelvey and Palfrey (1998) who use the quantal response equilibrium to analyse the Centipede game. In the quantal response equilibrium, players use a probabilistic choice function which leads to them making small errors, and account for the fact that other players may make errors when forming their beliefs. In equilibrium the probabilities determining the expected utilities of different strategies equal the probabilities with which players choose a specific strategy.[3] In contrast, given their beliefs, players do not make errors in an analogy-based expectations equilibrium.

[1] In contrast, in an analogy-based expectations equilibrium players must have consistent beliefs.

[2] The pure strategy analogy-based expectations equilibria are derived in Jehiel (2005).

[3] The solution concept is that of sequential equilbrium as these errors are not realised until a player actually moves.

Machine Games

The idea of using machine strategies to model players in finite horizon problems was introduced in the context of the prisoner's dilemma by Neyman(1985).[4] Rubinstein (1998) provides a general discussion of this approach. The underlying motivation of these models is that the complexity of a strategy can be represented by the number of states in the machine necessary to implement it. Players have effectively one decision, to choose a machine at the start. Strategies such as *always pass* and *always take* can be implemented without counting, and so need only one state. However, a strategy such as *take in the nth turn* requires a machine with at least n states to identify the nth period and implement it. If players have limited ability to count, or counting is costly (represented by a bounded number of states or a cost of additional states) then the strategy *take in the final node* may be costly for the second player to implement. If these costs or limits are significant, *always pass* may be superior and the cooperative outcome occurs. If the costs or limits are not significant then both players choose machines which *always take*, leading to the subgame-perfect equilibrium outcome. Therefore simple machine games cannot generate the strategy in which player 2 takes in the final node, nor the periods of mixing proposed for robust mixed strategy analogy-based expectations equilibrium in the Centipede game. In games such as the Doubling Dollar game the only equilibrium in simple strategies corresponds to the subgame-perfect equilibrium.

Other Approaches

Other related concepts of bounded rationality include Radner's (1980) notion of perfect ε equilibrium. This is applied to a finitely repeated Cournot game, in which firms use trigger strategies providing deviating does not confer a benefit greater than ε given the opponent's strategy. This reflects the idea that it is somehow costly to discover and implement the true best response. Although similar to the idea of optimality based refinement proposed in this thesis, the spirit behind it is quite different. Radner permits deviations to increase the set of equilibria and allow a cooperative outcome, while this thesis uses a related approach to reduce the set of mixed strategy analogy-based expectations equilibria. If $\varepsilon \geq 2$ then *always pass* is a pure strategy equilibrium for both players in the Centipede game (as player 2 does not have a sufficient incentive to change his strategy) but not the Doubling Dollar game.[5]

[4] In the protocol proposed by Neyman (1985) players use to send messages at the start of the game in a repeated prisoner's dilemma is not appropriate in the class of games considered in this paper, as *taking* ends the game.

[5] In Centipede like games this interpretation, which corresponds more closely to Radner's alternative definition of ε-equilibrium (see Radner, 1980) seems more natural.

This literature review has focused on how bounded rationality has been used to explain cooperation in finite horizon paradoxes. There are a number of other approaches which could be used. Particularly relevant to the Centipede game is the literature on psychological games of which Dufwenberg (2006) provides an excellent review. Psychological games allow utility functions to include beliefs. This is necessary when modelling reciprocity, in order to deduce the motivation behind an opponent's actions, so a player knows whether to respond with kindness or unkindness (Rabin, 1993). Another approach is that of learning, or machine learning. Ponti (2000) shows that eventually the Centipede game converges to the backwards induction solution if players are represented by continuous-time monotonic selection dynamics. Although he finds that this solution may be unstable, this is partly linked to the specification he chooses for the Centipede game, which in the notation in this thesis means that $\widehat{p} = \widehat{q} = \frac{6}{7}$ and even $\widehat{p} = \widehat{q} = \frac{60}{63}$. When players optimally *pass* even if they believe their opponent takes with up to 95% probability, it is not surprising that such dynamics may take a long time to converge to the subgame-perfect equilibrium, if the starting strategies are far from equilibrium and when there is a reasonably large chance of errors (drift in this class of models). However, the paper provides a convincing argument supporting backwards induction when players observe their opponent's strategies.

Chapter 13

Conclusion

An analogy-based expectations equilibrium (Jehiel, 2005) involves players bundling nodes at which their opponents move into analogy classes, and forming expectations that the opponents behave in the same way within each class. At every node players choose a best response to their beliefs and expectations are consistent with the *average* behaviour within an analogy class. The contribution of this thesis has been to investigate which analogy-based expectations equilibria are robust when players form analogy classes *endogenously*. The underlying idea is that players form their analogy classes from the same long history of past games in which they learn to have consistent analogy-based expectations. Players are less likely to form analogy classes over nodes in which the opponent's behaviour is very different, and form analogies more carefully when suboptimal actions could prove very costly.

Formal analysis was carried out on a class of games of complete and perfect information, intended as a generalisation of the Centipede game, in which there is a *finite horizon paradox*. Although it seems reasonable for players to *pass* for some periods when the game is long (and such behaviour is supported experimentally) this class of games has a unique subgame-perfect equilibrium in which players *take* at every node. Several approaches attempt to resolve this, including models of bounded rationality which are discussed in Chapter Twelve. In this class of games the analogy approach is particularly relevant because the decision faced by the opponent at each node is very similar, and the problem facing a player would be very simple *if* he knew his opponent's strategy. Complexity arises because the opponent's exact strategy is unknown, and therefore focusing on a method which simplifies beliefs directly, rather than simplifying strategies, is particularly appropriate.

In any analogy-based expectations equilibrium involving only pure strategies, one player must observe his opponent *take* with probability 1 at a node in which he expected the opponent to *pass* with high probability.[1] It was argued that the introduction of mixed strategy analogy classes has two effects which are likely to make an analogy-based expectations equilibrium more robust. Firstly, taking with probability 1 (which is inevitable at some point) is closer to mixing than it is to passing. Secondly, although the equilibrium still involves one of the players taking with certainty at some point, this node is only be observed with a relatively low frequency. Finally, mixing behaviour could be considered more complex to evaluate than pure strategies, so the formation of analogy classes is perhaps more natural when the opponent uses mixed strategies.

Introducing the idea of robustness serves the joint purpose of reducing the set of mixed strategy equilibria and formally capturing the idea that players form their analogy classes endogenously. Two methods of refinement were proposed, based on restricting the variation of behaviour permitted within an analogy class, and putting an upper bound on the suboptimality of a player's actions resulting from an analogy class. This type of optimality based refinement allows the robustness of the analogy-based expectations equilibrium to be linked to the underlying parameters which specify the game. In games in which a large part of the increase in payoffs is additive (with a high parameter μ), such as the Centipede game, cooperative equilibria were found to be more robust to optimality based refinement than games where the scaling of payoffs (represented by parameter λ) creates a much stronger incentive to be the first player to *take* (such as the Doubling Dollar game).

Although the two approaches to refinement are separate (and have different domains) it was shown that they are closely related and lead to very similar restrictions on behavioural strategies. In addition, mixed strategies may be dramatically more robust than pure strategies involving passing. For example, the consistency and optimality based refinement measures for the most robust mixed strategy equilibrium, derived in Examples 10.2 and 10.7, were $\theta_1 = 0.0060$ and $t_1 = 0.0079$ when $N = 4$. To achieve the same degree of payoff robustness, the passing equilibrium would need a Centipede game with 126 moves for each player; the same degree of consistency based robustness would require $N = 333$. This intuition was extended to the case when players form multiple analogy classes. For a given level of sophistication, a repeated game with many nodes has multiple combinations of robust analogy classes, which could be thought as representing different degrees of coordination between the players. The approach proposed in this thesis provides an intuitive solution to the finite horizon problem: an equilibrium consists of players passing for a given number of nodes and

[1] Except the unique subgame perfect Nash equilibrium.

then mixing towards the end of the game. Such a solution is illustrated in Figure 11.1, which shows a top down view of the Centipede game, where at every node taking is equivalent to playing down. In this example, each player forms two analogy classes. In Ω_1 and Ψ_1 players expect their opponent to *pass*, while in Ω_2 and Ψ_2 players expect their opponent to mix with probabilities which leave them indifferent. A promising extension of this approach was discussed in Section 11.4, which proposed a generalisation of the solution concept, relaxing the consistency requirement so that the mean absolute deviation (or variance) of behaviour from $\widehat{p}$ must be less than some constant η. This analysis addresses the idea that it is strategic uncertainty which leads to complexity in forming beliefs, in contrast to a difficulty in optimising when players form beliefs correctly.

Part III

Common Value Multi-Unit Auctions

Disclaimer: This part of the thesis is a revised, restructured and extended adaptation of my MPhil thesis, titled "*IPO Auctions*" and intended as a more applied paper. As well as this change in focus, Chapters 16 and 17 formalise proofs that were only sketched in the original. Previously, many equilibria were omitted and the proposed equilibria of the Vickrey and Uniform auctions in Chapters 18 (and some in Chapter 17) were incorrect (the former constitutes a major result of this thesis). As well as these corrections, Chapter 18 also extends the analysis of the discriminatory auction to compare it more completely to the analysis of the auction for the bundle. Chapter 19 and almost all of Chapter 20 is entirely new.

Chapter 14

Introduction

Although the understanding of auctions where bidders have multi-unit demand is growing, most of the literature focuses on the cases in which bidders have either independent private valuations or constant marginal valuations for all additional units. This thesis explores the implications for efficiency and expected revenue of using different multi-unit auction mechanisms when bidders have multi-unit demand and a common valuation V^m for the mth object they win. The information and payoff structure is that bidders have independently distributed signals and value the first object they win at the average of bidders' signals.[1] It is assumed for simplicity that there are two bidders and two objects for auction, so for bidder i, $V_i^1 = \frac{1}{2}(s_i + s_j)$. The extent to which the results generalise will be discussed at the end of the thesis.

This analysis shows that unlike the case when bidders have independent private values, common value auctions may decompose into multiple single-unit auctions in equilibrium because of the way bidders adjust their strategies to allow for the winner's curse. When bidders have increasing or decreasing marginal valuations for additional units, these auctions are asymmetric, so insights from the analysis of asymmetric auctions can be used to analyse the discriminatory and Vickrey auctions. It is shown that revenue equivalence holds when bidders have constant or increasing marginal valuations, although the uniform price auction has an additional *demand reduction* equilibrium which is inefficient and generates low revenue; in addition, all of the auctions considered have equilibria which involve bidders submitting flat demand curves ($b_i^1 = b_i^2$), which are not supported empirically. This motivates the introduction of decreasing marginal valuations in Chapter Eighteen, in which case second-price auctions are efficient but yield low revenues. The discriminatory auction raises the most revenue of all the auctions considered, including auctioning the bundle, although it is

[1] This is a modified version of the wallet game proposed by Klemperer (1998).

not generally efficient.

Chapter Twenty-One will consider some empirical results from experimental economics and treasury auctions. While some of the empirical results are explained by introducing more units and bidders, others, such as overbidding on a first unit, can only be explained by introducing bounded rationality, which seems reasonable as the complexity and uncertainty of the situation make it harder to condition on winning and to calculate optimal bidding strategies. Even if most bidders are able to fully comprehend the multi-unit auction, the presence of a few smaller bidders who are boundedly rational may introduce supply uncertainty. Generalisation of the model will show that expected revenue in the uniform price auction is non-monotonic as the number of bidders and units changes; this is reinforced by bounded rationality as in some cases implicitly collusive equilibria can be supported by simple bidding strategies.

14.1 Independent Private Value Auctions

Traditionally the theory of single-unit auctions has focused on first- and second-price, sealed-bid auctions, ascending auctions and descending auctions. Assuming that bidders are risk neutral and have independent private valuations, Vickrey (1961) showed that these four different mechanisms generate the same expected revenue, equal to the expected valuation of the second highest bidder. The intuition behind this is briefly analysed, as the revenue equivalence theorem applies to the case when the objects are sold in a bundle, either explicitly by the auctioneer or implicitly as a feature of the equilibrium.

In the first-price, sealed-bid auction each bidder submits a bid without observing the bid of any other player; the highest bid wins the auction and the winning bidder pays a price equal to this bid. This is strategically equivalent to the descending auction in which the auctioneer continuously lowers the price until a bidder takes the item at its current price. Each bidder faces a trade off; raising their bid increases the chance of winning, but lowering it increases the surplus they receive if they do win. Each bidder's strategy is a function of his own private valuation and his prior beliefs about the valuations of the other bidders. If there are N bidders each with valuation s_i drawn independently from a uniform distribution over $[0\ ,\ 1]$ then there is a unique symmetric Bayesian Nash equilibrium in which each player uses a bidding strategy

$b_i(s_i) = \frac{N-1}{N} s_i$. Therefore the expected revenue raised is:

$$\begin{aligned} E\left[\max_i (b_i(s_i))\right] &= E\left[\max_i \left(\frac{N-1}{N} s_i\right)\right] = \frac{N-1}{N} E\left[\max_i (s_i)\right] \\ &= \frac{N-1}{N} \frac{N}{N+1} = \frac{N-1}{N+1} \end{aligned}$$

This is equal to the expected valuation of the second highest bidder,

$$E[\text{second highest } s_i] = \frac{N-1}{N+1}$$

In an ascending auction each bidder can raise the current price. The auction ends when no bidder will raise the price further and the last bidder to raise the price wins the auction and pays the current price. In this mechanism it is a dominant strategy for bidders to raise the current price until their own valuation is reached. Thus the bidder with the highest valuation wins the auction and pays the price at which the second highest bidder drops out, meaning that the revenue raised equals the actual valuation of the second highest bidder.

In the Vickrey auction (second-price, sealed-bid in the single-unit case) each bidder submits a bid without observing the bids of any other players. The highest bid wins the auction but the winning bidder pays the price of the second highest bid. It is a weakly dominant strategy for players to bid their true valuations. If they bid above their valuation, then reducing their bid only reduces the probability of winning in situations where they would receive a negative surplus. If they bid below their valuation then increasing their bid increases the probability of them winning without reducing the surplus they receive if they win. Therefore the bidder with the highest valuation wins the object and the revenue raised is equal to the second highest bidder's valuation.

14.2 Common Value Auctions

In common value auctions winning provides the same utility to all bidders, but initially they have incomplete information of this value. Rational bidders should adjust their strategies to avoid the winner's curse. This is an empirical phenomenon used to describe the case when bidders fail to realise that winning implies bad news about the valuations of other players, reducing their own expected valuation conditional on winning the unit. In equilibrium, fully rational bidders avoid the winner's curse by basing their bids on the value of the object *conditional on winning it* rather than on an unconditional expected valuation.

If there are three or more bidders with affiliated signals,[2] so that a higher realisation for one bidder makes a higher realisation for another more likely, then the revenue equivalence theorem breaks down because of the linkage principle. This is because in second-price auctions, the expected price paid by a bidder is now increasing in his signal (as the second highest bid is positively correlated with his signal) so bidders with high signals pay more. This is not the case in first-price auctions and therefore the expected revenue from a second-price auction is greater than that for a second-price auction. In the *wallet game* bidders have common values but independent signals, so in the symmetric case the revenue equivalence theorem holds.

Clearly other factors affect the practical choice of an auction. For example, if there are few bidders then a first-price auction might be more robust to collusion, as it provides bidders with a strong incentive to cheat on any pre-agreed bidding strategies. Secondly, small asymmetries can have a very large effect on equilibrium strategies and revenues in second-price auctions, as the winner's curse is greatly magnified for weaker bidders, so a first-price auction might be chosen to maximise expected revenue. Other factors such as the presence of budget constraints or when entry is endogenous might also affect the choice of auction format.

14.3 Multi-Unit Auctions

All sealed-bid multi-unit auctions considered in this thesis have the same allocation rule. Bidder i submits k marginal bids b_i^k. The auctioneer aggregates all the bids to form a demand curve, and awards a unit to the K highest bids. This reordering of bids constrains bidders to submit weakly decreasing demand curves, so when there are two units $b_i^{k+1} \geq b_i^k \ \forall \ (k\ ,\ s\ ,\ i)$. This does not affect the equilibrium outcomes developed in this thesis.[3] For example, when there are two units in the auction if bidder i submitted bids of (0.6, 0.3) and bidder j submitted bids of (0.4, 0.1), each bidder would be allocated a single-unit. This example is illustrated in Figure 14.1.[4]

[2] Milgrom and Weber (1982).

[3] When bidders have constant or diminishing marginal valuations, it is optimal to submit weakly decreasing demand curves. When bidders have increasing marginal valuations, in equilibria of the common values model developed in this paper, typically both units will be awarded to the same bidder, so it is $b_i^1 + b_i^2$ which is important, not the individual bids. This will be discussed in more detail at the end of Chapter 6.

[4] The auctioneer satisfies the highest marginal bids first, effectively constraining each bidder's strategy space so that $b_i^1 > b_i^2$ because of the way the auctioneer compiles them.

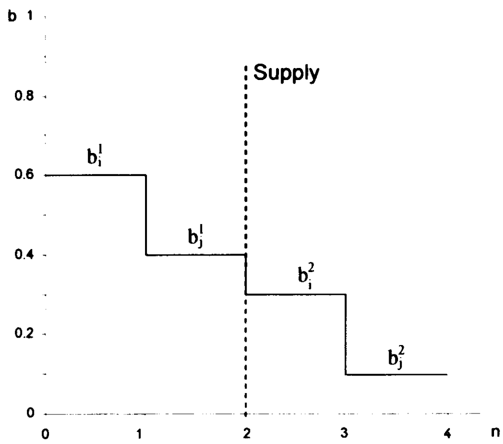


Figure 14.1: Aggregated bid functions

Figure 14.1 is also useful for illustrating different payment rules in the auction. In the discriminatory auction, bidders pay their winning bids for each unit they are awarded, so in the example above, bidder i pays a total bid of $b_i^1 = 0.6$ and bidder j pays $b_j^1 = 0.4$. If there is only one unit being auctioned then the discriminatory auction is equivalent to a first-price, sealed-bid auction.

In the uniform price auction, each bidder pays the same price for every unit won. Consistent with the literature, this thesis follows Friedman (1960) and develops explicitly the uniform price auction where the price equals the highest losing bid. Alternatively, it could equal the lowest winning bid or some price between these bids. Assuming bidders pay the highest losing bid then in the example above both i and j pay $b_i^2 = 0.3$, as each wins one unit. In the single-unit sealed-bid second-price auction it is a weakly dominant strategy for each bidder to bid his valuation, but in the multi-unit case this truthfulness property is lost. There is a chance that any bid after the first will become the marginal bid that determines the price paid for all units won, so bidders engage in *demand reduction*, reducing bids on later units to reduce the expected price they will pay for earlier units, and therefore increasing their surplus.

In the Vickrey auction for multiple units,[5] bidders pay the auctioneer's opportunity cost for each unit won. For bidder i this equals the sum of the bids which would have won a unit if bidder i had not bid in the auction. In the example illustrated in Figure 14.1 bidder i pays b_j^2 and bidder j pays b_i^2. In the case of independent private valuations the Vickrey auction gives bidders the incentive to bid their valuations. In

[5] In this paper, "Vickrey auction" refers to the multi-unit Vickrey auction. The second-price sealed-bid auction for a single unit is referred to as a second-price auction.

the case of common values and diminishing marginal valuations, this thesis shows that the Vickrey auction may be undesirable. In addition, Ausubel and Milgrom (2002) show that there may be monotonicity problems with the Vickrey auction when the objects are not substitutes, as additional bidders may lower expected revenue. The assumption of common values and symmetry between bidders assumed in this thesis avoids this problem, as implicit bundling occurs when bidders have increasing marginal demands.[6]

In general, intramarginal bids submitted in the discriminatory auction are generally lower, as bidders aim to pay just above the clearing price. This is less important in a uniform price auction as each bidder pays the same regardless of what they bid. Much of the literature relating to treasury auctions is aimed at determining whether uniform price or discriminatory auctions raise more revenue. In the discriminatory auction, several papers show flat demand curves occur in equilibrium (see Back and Zender, 1993, Lebrun and Tremblay, 2003 and Ausubel 2004). This motivates the introduction of decreasing marginal valuations. In uniform price auctions, the existence of low revenue equilibria is well established (see Wilson, 1979, Back and Zender, 1993, Engelbrecht-Wiggans and Khan, 1998). The contribution of this paper is to analyse decreasing marginal valuations and show that this can lead to the low revenue equilibrium being unique. Secondly it considers the introduction of additional bidders and units, and argues that as bidding strategies are simple when the number of units is perfectly divisible by the number of bidders, bounded rationality may support significant non-monotonicities.

[6]When units are complements bidders would choose to bid $b_i^2 > b_i^1$ in equilibrium when they have increasing marginal valuations.

Chapter 15

The Model

The model attempts to capture the strategic implications of an auction in which bidders have multi-unit demand for objects and a common value V^m for the mth unit they win. The information and payoff structure is such that bidders have independently distributed signals and value for the first object they win at the average of bidder's signals.[1] It is assumed for simplicity that there are two bidders and two objects for auction, so for bidder i, $V_i^1 = \frac{1}{2}(s_i + s_j)$. Signals are uniformly distributed over the range $[0\ ,\ 1]$. The aim of this analysis is to develop a simple model which is tractable for a range of different valuation scales. The extent to which the model generalises will be discussed at the end of the thesis. In addition, it will be shown that introducing bounded rationality may mitigate some of the extreme theoretical results while reinforcing others. This is a particularly relevant consideration in multi-unit auctions, which are often complex.

Although bidders have a common value V^m for the mth unit they win, they may not have constant marginal valuations for every unit. Both bidders derive the same utility $\frac{1}{2}(s_i + s_j)$ from winning a single-unit and $\frac{1+\alpha}{2}(s_i + s_j)$ from winning two units. The subsequent analysis will focus on three cases. When $\alpha = 1$ bidders have constant marginal valuations so $V_i^1 = V_i^2 = \frac{s_i+s_j}{2}$. When $\alpha < 1$ bidders have diminishing marginal valuations and $V_i^2 < V_i^1$. Finally there may be situations when bidders have increasing marginal valuations, for example when winning a second unit gives a bidder monopoly power. In this final case $\alpha > 1$ and $V_i^2 > V_i^1$.

Bidder i submits a marginal bid of b_i^m for each unit m. Inverse bid functions are defined such that $\phi_i^m(b_i^m)$ is the inverse of $b_i^m(s_i)$, so $b_i^2(s_i)$ gives player i's bid for a second unit (as a function of s_i) and the inverse $\phi_i^2(b)$ determines the signal necessary for bidder i to submit a bid of b for the second unit. The allocation rule is the same in

[1] This is a modified version of the wallet game proposed by Klemperer (1998).

all the auctions considered, with the two highest marginal bids each receiving one unit. This constrains bidders to submitting downward sloping demand curves, as $b_i^1 \geq b_i^2$. Allowing bidders to submit specific bids for their first and second units does not affect the equilibria which are developed explicitly in this thesis.[2]

In the formal analysis, bidders rationally compensate for the winner's curse by conditioning their expectations of the value of the objects on the possible outcomes.[3] If $b_j^1 > b_j^2$ then the expectation of s_j is lower the more units bidder i wins. As $b_i^1 \geqq b_i^2$ for both bidders, the outcome for bidder i can be broken down into three cases:

Case 1: Bidder i wins two units	$b_i^2 > b_j^1 \Rightarrow s_j < \phi_j^1\left(b_i^2\right)$
Case 2: Bidder i wins one unit	$b_j^1 > b_i^2 \Rightarrow s_j > \phi_j^1\left(b_i^2\right)$
	$b_i^1 > b_j^2 \Rightarrow s_j < \phi_j^2\left(b_i^1\right)$
$\Rightarrow$	$\phi_j^1\left(b_i^2\right) < s_j < \phi_j^2\left(b_i^1\right)$
Case 3: Bidder i wins zero units	$b_j^2 > b_i^1 \Rightarrow s_j > \phi_j^2\left(b_i^1\right)$

Table 15.1: Allocations Conditional on s_j

This is summarised in Figure 15.1 which shows how different allocations depend on the realisation of s_j. As s_j is uniformly distributed the probability of winning a given number of units is represented by the length of the line between thresholds, and the expected value of s_j conditional on winning a given number of units is given by the midpoint of a section of the line. Figure 18.1 gives a specific example of how this is derived for the discriminatory auction when $\alpha = \frac{3}{5}$.

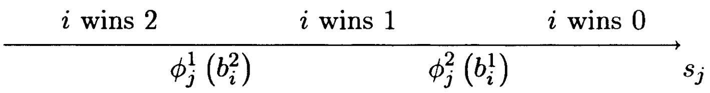


Figure 15.1: Allocations conditional on s_j

If in equilibrium $\phi_j^1\left(b_i^2\right) \leqslant \phi_j^2\left(b_i^1\right)$ and $\phi_i^1\left(b_j^2\right) \leqslant \phi_i^2\left(b_j^1\right)$ always binds then implicit bundling occurs, as both units are always allocated to the same player.

The probabilities and expectations are summarised on the following page. To avoid complicating the notation further, all of the optimisation carried out by bidder i is conditional on the signal s_i. *This notation is omitted, but unless otherwise stated all*

[2] This is because when $b_i^1 \geq b_i^2$ binds, it does so for all bidders and all signals, leading to implicit bundling. This will be discussed in more detail in Chapter Seventeen.

[3] Chapter Twenty-One will discuss the impact of bidders failing to fully compensate for the winner's curse.

expectations and probabilities in this thesis are conditional on a bidder's signal. In addition bid functions are generally written without including signals.

Probability of bidder i winning one unit only:	$\Pr(win1) = \phi_j^2(b_i^1) - \phi_j^1(b_i^2)$
Probability of bidder i winning two units:	$\Pr(win2) = \phi_j^1(b_i^2)$
Expectation of s_j conditional on bidder i winning one unit:	$E[s_j \mid win1] = \frac{\phi_j^2(b_i^1)+\phi_j^1(b_i^2)}{2}$
Expectation of s_j conditional on bidder i winning two units:	$E[s_j \mid win2] = \frac{\phi_j^1(b_i^2)}{2}$

The allocation rule is to award a unit to each of the two highest marginal bids, so these probabilities apply to all the different auction formats considered in this thesis. However, the expected payments will depend on the type of auction used. Bidder i aims to maximise expected surplus:

Net payoff to bidder i for winning one unit:	$\frac{s_i+s_j}{2} - p_i^1$
Net payoff to bidder i for winning two units:	$\frac{1+\alpha}{2}(s_i+s_j) - p_i^2$

For example, in the discriminatory auction bidders pay their own bids, so the price paid when winning one unit is b_i^1, and the price paid conditional on winning two units is $b_i^1 + b_i^2$. Bidder i aims to maximise expected surplus:

$$\begin{aligned} U_i &= E[surplus] \\ &= E\left[\frac{s_i+s_j}{2} - p_i^1 \mid win1\right]\Pr(win1) \\ &\quad +E\left[\frac{1+\alpha}{2}(s_i+s_j) - p_i^2 \mid win2\right]\Pr(win2) \end{aligned} \tag{15.1}$$

This is because the events of winning either one or two units are mutually exclusive and the expected surplus conditional on winning no units is zero in the auction formats considered here.

Chapter 16

Decomposition and First Order Conditions

This section demonstrates that the discriminatory and Vickrey auctions decompose into two single-unit auctions when the requirement that $b_i^1 \geq b_i^2 \ \forall \ (s\ ,\ i)$ does not constrain the optimal bids in equilibrium. The intuition is that for either bidder to win a second unit, he competes against his opponent's bid for a first unit. This section will also develop the first order conditions for the uniform price auction and derive the first-price, sealed-bid auction for the bundle of units. Later sections will show that these first order conditions form part of an equilibrium, allowing the efficiency and revenues of these different auction mechanisms to be compared.

16.1 The Discriminatory Auction

In the discriminatory auction bidder i pays a bid of b_i^1 conditional on winning one unit and $b_i^1 + b_i^2$ conditional on winning two units. The expected surplus can be calculated by substituting $p_i^1 = b_i^1$ and $p_i^2 = b_i^1 + b_i^2$ into Equation 8.1, as well as the probabilities and conditional expectations of s_j in Table 15.1.

$$\begin{aligned} U_i &= E\left[\frac{s_i + s_j}{2} - b_i^1 \mid win1\right] \Pr(win1) \\ &\quad + E\left[\frac{1+\alpha}{2}(s_i + s_j) - (b_i^1 + b_i^2) \mid win2\right] \Pr(win2) \\ &= \frac{1}{4}\left[\begin{array}{c} \left(2s_i + \phi_j^1\left(b_i^2\right) + \phi_j^2\left(b_i^1\right) - 4b_i^1\right)\left[\phi_j^2\left(b_i^1\right) - \phi_j^1\left(b_i^2\right)\right] \\ + \left(2\left(1+\alpha\right)s_i + \left(1+\alpha\right)\phi_j^1\left(b_i^2\right) - 4(b_i^1 + b_i^2)\right)\phi_j^1\left(b_i^2\right) \end{array}\right] \end{aligned}$$

Differentiating U_i with respect to $b_i^1$and b_i^2 gives the following first order conditions:

$$\begin{aligned}\frac{\partial U_1}{\partial b_i^1} &= \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2'}\left(b_i^1\right) - \phi_j^2\left(b_i^1\right)\\ \frac{\partial U_1}{\partial b_i^2} &= \left[\frac{\alpha}{2}\left[s_i + \phi_j^1\left(b_i^2\right)\right] - b_i^2\right]\phi_j^{1'}\left(b_i^2\right) - \phi_j^1\left(b_i^2\right)\end{aligned}$$

The first order conditions for bidder j are the same due to symmetry.

Proposition 16.1 *When bids are increasing, continuous functions of signals the discriminatory auction decomposes into multiple first-price, sealed-bid single-unit auctions providing that $b_i^1 \geq b_i^2\ \forall\ (s\ ,\ i)$ does not constrain the optimal bids in equilibrium.*

Proof. The first order conditions for b_i^1 and b_j^2 are:

$$\begin{aligned}b_i^1 &: \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2'}\left(b_i^1\right) = \phi_j^2\left(b_i^1\right)\\ b_j^2 &: \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right]\phi_i^{1'}\left(b_j^2\right) = \phi_i^1\left(b_j^2\right) \qquad (16.1)\end{aligned}$$

Assuming that the condition $b_i^1 \geq b_i^2\ \forall\ (s\ ,\ i)$ does not constrain the optimal bids, in equilibrium b_i^1 and b_j^2 can be solved independently of b_j^1 and b_i^2, as neither these bids nor their inverses enter these first order conditions. This is equivalent to the equilibrium of an asymmetric first-price, sealed-bid auction for a single-unit. The same applies to bids b_i^2 and b_j^1 by symmetry. ■

The intuition behind this result is that bidder i's bid for a first unit competes against bidder j's bid for a second unit. This decomposition relies on the assumption that it is not optimal to set $b_i^2 > b_i^1$ in the equilibria of these single-unit, first-price auctions, so in the multi-unit auction $b_i^1 \geq b_i^2$ does not constrain the optimal strategies for either player. It will be shown that the assumption of common values, leading to the winner's curse, means that in equilibrium $b_i^1 \geq b_i^2$ constrains the optimal strategies either at all signals or none, depending on the parameter α. If $\alpha \geq 1$, it always binds so $b_i^1 = b_i^2\ \forall\ (s\ ,\ i)$ and the discriminatory auction results in implicit bundling, as both units will always be allocated to the same bidder. If $\alpha \leq 1$ then $b_i^1 \geq b_i^2$ does not constrain the optimal bids in equilibrium, and the discriminatory auction decomposes into multiple first-price, sealed-bid single-unit auctions. It is worth noting that when $\alpha = 1$, both of these statements hold, so the constant marginal valuations most often dealt with in the literature is actually a special case where the auction both decomposes *and* leads to implicit bundling.

16.2 The Vickrey Auction

In the Vickrey auction bidders pay the opportunity cost of their bid to the auctioneer. When there are two bidders this is the second highest bid for each unit they win.[1] Therefore if bidder i wins a single-unit he pays b_j^2, bidder j's bid for a second unit. If bidder i wins both units then he pays $\left(b_j^1 + b_j^2\right)$. Bidder i is not aware of j's bids but can form the following expectations conditional on winning either one or two units.

Expected bid paid conditional on bidder i winning one unit:
$$E\left[b_j^2 \mid win1\right] = \frac{1}{\phi_j^2(b_i^1)-\phi_j^1(b_i^2)} \int_{\phi_j^1(b_i^2)}^{\phi_j^2(b_i^1)} b_j^2(x)dx$$

Expected total bid paid conditional on bidder i winning two units:
$$E\left[b_j^1 + b_j^2 \mid win2\right] = \frac{1}{\phi_j^1(b_i^2)} \int_{0}^{\phi_j^1(b_i^2)} b_j^1(x) + b_j^2(x)dx$$

These bids can be substituted into the expression for expected surplus derived in Equation 15.1:

$$\begin{aligned}
U_i &= E\left[\frac{s_i+s_j}{2} - b_j^2 \mid win1\right] \Pr(win1) + E\left[\left(\frac{1+\alpha}{2}\right)(s_i+s_j) - b_j^1 - b_j^2 \mid win2\right] \Pr(win2) \\
&= \frac{1}{4}\left[\begin{array}{c} \left[2s_i + \phi_j^2\left(b_i^1\right) + \phi_j^1\left(b_i^2\right)\right]\left(\phi_j^2\left(b_i^1\right) - \phi_j^1\left(b_i^2\right)\right) \\ -4\displaystyle\int_{\phi_j^1(b_i^2)}^{\phi_j^2(b_i^1)} b_j^2(x)dx + (1+\alpha)\left[2s_i + \phi_j^1\left(b_i^2\right)\right]\phi_j^1\left(b_i^2\right) - 4\displaystyle\int_{0}^{\phi_2^1(b_1^2)} b_j^1(x) + b_j^2(x)dx \end{array}\right]
\end{aligned}$$

Differentiating U_i with respect to b_i^1 and b_i^2 gives the following:

$$\begin{aligned}
\frac{\partial U_i}{\partial b_i^1} &= \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2\prime}\left(b_i^1\right) \\
\frac{\partial U_i}{\partial b_i^2} &= \left[\frac{\alpha}{2}\left[s_i + \phi_j^1\left(b_i^2\right)\right] - b_i^2\right]\phi_j^{1\prime}\left(b_j^2\right)
\end{aligned}$$

Proposition 16.2 *When bids are continuous, increasing functions of signals the multiple unit Vickrey auction decomposes into multiple second-price, sealed-bid single-unit auctions, providing that $b_i^1 \geq b_i^2 \ \forall \ (s\ ,\ i)$ does not constrain the optimal bids in equilibrium.*

Proof. The first order conditions for b_i^1 and b_j^2 can be calculated as follows:

$$\begin{aligned}
b_i^1 &: \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2\prime}\left(b_i^1\right) = 0 \\
b_j^2 &: \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right]\phi_i^{1\prime}\left(b_i^2\right) = 0 \qquad (16.2)
\end{aligned}$$

[1] In this simple case of two bidders and one-dimensional signals the Vickrey auction is straightforward to implement. The problems which arise in these other cases will be discussed at the end of this part of the thesis.

Assuming that the condition $b_i^1 \geq b_i^2$ does constrain the optimal bids, the equilibrium bids b_i^1 and b_j^2 can be solved independently of b_j^1 and b_i^2, as neither these bids nor their inverses enter these first order conditions. This is equivalent to the equilibrium of an asymmetric second-price, sealed-bid auction for a single-unit. An equivalent result can be derived for b_i^2 and b_j^1 by symmetry. ∎

The intuition used to motivate the decomposition of the discriminatory auction also applies to the Vickrey auction. Bidder i's bid for a first unit competes against bidder j's bid for a second unit. This decomposition depends on the assumption that it is not optimal to set $b_i^1 < b_i^2$ in the equilibria of these single-unit, second-price auctions, so in the multi-unit auction $b_i^1 \geq b_i^2$ does not constrain the optimal strategies for either player. As with the discriminatory auction, it will be shown that the requirement $b_i^1 \geq b_i^2 \ \forall \ (s \ , \ i)$ constrains the optimal strategies either at all signals or none, depending on the parameter α, so either the Vickrey auction decomposes into two single-unit auctions or it results in implicit bundling, as both units will always be allocated to the same bidder. When implicit bundling occurs the Vickrey auction is equivalent to the single-unit, second-price, sealed-bid auction for the bundle of units.

16.3 The Uniform Price Auction

In the uniform price auction the same price is paid for every unit won. In the case developed here explicitly this is the highest losing bid. As $b_j^1 \geq b_j^2 \ \forall \ s$, if bidder i wins two units he pays a price of $2b_j^1$. If each bidder wins one unit then they both pay $Max\left(b_i^2 \ , \ b_j^2\right)$. Bidder i will not know j's bids in advance, but can form expectations of the price conditional on winning either one or two units. The full derivation of this auction is in Appendix C.2, giving the following differentials for bidder i:

$$\begin{aligned}
\frac{\partial U_i}{\partial b_i^1} &= \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2\prime}\left(b_i^1\right) \\
\frac{\partial U_i}{\partial b_i^2} &= \left[\frac{\alpha}{2}\left[s_i + \phi_j^1\left(b_i^2\right)\right] - b_i^2\right]\phi_j^{1\prime}\left(b_i^2\right) + \phi_j^1\left(b_i^2\right) - \phi_j^2\left(b_i^2\right)
\end{aligned}$$

Proposition 16.3 *If there is an implicit bundling equilibrium of the Vickrey auction in which $b_i^1 = b_i^2 \ \forall \ (s \ , \ i)$, this is also an equilibrium of the highest losing bid uniform price auction.*

Proof. The first order conditions for b_i^1 and b_j^2 in the highest losing bid uniform price auction are:

$$\begin{aligned}
b_i^1 &: \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2\prime}\left(b_i^1\right) = 0 \\
b_j^2 &: \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right]\phi_i^{1\prime}\left(b_j^2\right) = \phi_i^2\left(b_j^2\right) - \phi_i^1\left(b_j^2\right) \qquad (16.3)
\end{aligned}$$

If $b_i^1(s) = b_i^2(s) \; \forall \; s \; , \; i$ is an equilibrium of the Vickrey auction then it must be true that inverting these conditions $\phi_i^2(b) = \phi_i^1(b) \; \forall \; s \; , \; i$. Therefore $\phi_i^2\left(b_j^2\right) - \phi_i^1\left(b_j^2\right) = 0$ and the first order conditions for the uniform price auction are identical to those of the Vickrey auction. Chapter Seventeen will demonstrate that this constitutes an equilibrium. ■

In the highest losing bid uniform price auction the first order condition for a bid on the first unit is identical to that for the Vickrey auction. For the second unit, bidders submit bids weakly below those in the Vickrey auction. This is deduced by considering bidder j's bid for a second unit, b_j^2, as $b_i^1 \geq b_i^2 \; \forall \; (s \; , \; i)$ implies that $\phi_i^2(b) \geq \phi_i^1(b)$, so in general when $b_j^2 = \frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right]$, $\frac{\partial U_i}{\partial b_j^2} = \phi_i^1\left(b_j^2\right) - \phi_i^2\left(b_j^2\right) \leq 0$. This represents the effect of *demand reduction*, where bidders reduce their bid for a second unit to lower the expected price paid for the first unit. The extent of *demand reduction* depends on $\phi_i^1\left(b_j^2\right) - \phi_i^2\left(b_j^2\right)$ which is the probability that bidder j wins one unit and pays b_j^2. If the probability of this is 0, which occurs when $b_i^1 = b_i^2 \; \forall \; s_j$ there is no *demand reduction.* However, the next chapter will show that even when this is an equilibrium, there is another equilibrium in which *full demand reduction* generates a market clearing price of zero.

16.4 Bundling The Units

The auctioneer could bundle the units and sell them using a single-unit auction. The equilibrium of the first-price auction for the bundle is derived here for comparison with the discriminatory auction. Bidders have a common valuation for the bundle of $(1+\alpha)\frac{s_i+s_j}{2}$ and conditional on winning bidder i pays his own bid b_i. As $\Pr(b_i > b_j) = \Pr(\phi_j(b_i) > s_j)$ the probability of winning equals $\phi_j(b_i)$ and $E[s_j \mid win] = \frac{\phi_j(b_i)}{2}$ when $s_j \sim U[0 \; , \; 1]$. Therefore the expected surplus is:

$$\begin{aligned} U_i &= E\left[(1+\alpha)\frac{s_i+s_j}{2} - b_i \mid win\right]\Pr(win) \\ &= \left[(1+\alpha)\left[\frac{s_i}{2} + \frac{\phi_j(b_i)}{4}\right] - b_i\right]\phi_j(b_i) \end{aligned}$$

Differentiating U_i with respect to b_i and solving for b_j by symmetry gives the following first order conditions:

$$\begin{aligned} b_i &: \left[\frac{1+\alpha}{2}\left[s_i + \phi_j(b_i)\right] - b_i\right]\phi\prime_j(b_i) = \phi_j(b_i) \\ b_j &: \left[\frac{1+\alpha}{2}\left[s_j + \phi_i(b_j)\right] - b_j\right]\phi\prime_i(b_j) = \phi_i(b_j) \end{aligned} \tag{16.4}$$

Proposition 16.4 *The first-price auction for the bundle of units has a symmetric equilibrium in continuous strategies in which* $b_i = \frac{1+a}{2}s_i$ *and* $b_j = \frac{1+a}{2}s_j$. *This is efficient if* $a \geq 1$ *and inefficient if* $\alpha < 1$, *and raises an expected revenue for the auctioneer of* $\frac{1+a}{3}$.

Proof. It is routine to verify that these strategies constitute a local optima by substituting them into the first order conditions above. For example, if $b_j = \frac{1+a}{2}s_j$ then $\phi_j(b_i) = \frac{2}{1+\alpha}b_i$ and the first order condition for b_i becomes $\frac{1+\alpha}{2}s_i = b_i$. Secondly note that $\frac{\partial U_i}{\partial b_i^1} = \left[\frac{1+\alpha}{2}\left[s_i + \phi_j(b_i)\right] - b_i\right]\phi\prime_j(b_i) - \phi_j(b_i) = s_i - \frac{2}{1+\alpha}b_i$ and therefore $\frac{\partial U_i}{\partial b_i^1} > 0$ if $b_i < \frac{1+\alpha}{2}s_i$ and $\frac{\partial U_i}{\partial b_i^1} < 0$ if $b_i > \frac{1+\alpha}{2}s_i$. Therefore this strategy profile is globally optimal. The efficiency results follow from the argument that when $\alpha < 1$ bidders have diminishing marginal valuations and so bundling the units is inefficient. When $a \geq 1$ bundling is efficient, as bidders have constant or increasing marginal valuations. The expected revenue can be calculated as $E\left[R\right] = E\left[Max\left(\frac{1+a}{2}s_i\ ,\ \frac{1+a}{2}s_j\right)\right] = \frac{1+a}{2}E\left[Max\left(s_i\ ,\ s_j\right)\right] = \frac{1+a}{3}$. ■

When the auctioneer bundles the units and sells them in a single-unit auction, the implicit asymmetry which occurs when multi-unit auctions decompose does not arise, as both bidders have the same valuation for the bundle. As the allocation rules (and hence interim probabilities) and interim expected payments are the same in the symmetric equilibria of all the methods of auctioning the bundle, the revenue equivalence theorem holds so any standard auctions generates the same expected revenue.[2] Therefore it is the question of whether or not to bundle which is important, not how to auction the bundle.

Chapter Seventeen will show that if the discriminatory auction leads to implicit bundling then it is equivalent to the first-price, sealed-bid auction for the bundle. The bid paid for the bundle equals the sum of the marginal bids in the discriminatory auction, so $b_i = b_i^1 + b_i^2$. If the Vickrey auction results in implicit bundling then it is equivalent to the second-price, sealed-bid auction of the bundle. In this case if bidder i wins then he pays b_j, the opponent's bid for the bundle, which equals the sum of j's marginal bids in the Vickrey auction, $b_j = b_j^1 + b_j^2$. This thesis will argue that implicit bundling is likely when bidders have increasing or constant marginal valuations.

[2] In the common values setting the second price auctions will have additional asymmetric equilibria which will lead to lower revenues (Milgrom, 1981).

Chapter 17

Constant Marginal Valuations

The constant marginal valuations case is motivated on a liquid secondary market, meaning common values hold for all bidders and units. When $\alpha = 1$ any allocation is efficient as all marginal valuations are the same. The payoffs equal the average of the bidders' signals and are summarised as $V_i^1 = V_i^2 = \frac{s_1+s_2}{2}\ \forall\ i$.

17.1 The Discriminatory Auction

In a discriminatory auction Proposition 16.1 demonstrated that providing $b_i^1 \geq b_i^2\ \forall\ (s\ ,\ i)$ does not constrain the optimal bids in equilibrium, first order conditions for bids b_i^1 and b_j^2 are independent of b_j^1 and b_i^2. The case of constant marginal valuations is special as in equilibrium the constraint $b_i^1 \geq b_j^2$ binds exactly at the optimal bids. However, as it does not actually constrain the bids, the auction can still be solved by assuming it decomposes into separate single-unit, first-price auctions in an equilibrium which also involves implicit bundling.

$$\begin{aligned} b_i^1 &: \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right] \phi\prime_j^2\left(b_i^1\right) = \phi_j^2\left(b_i^1\right) \\ b_j^2 &: \left[\frac{1}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right] \phi\prime_i^1\left(b_j^2\right) = \phi_i^1\left(b_j^2\right) \end{aligned}$$

Proposition 17.1 *When bidders have constant marginal valuations the discriminatory auction has a symmetric equilibrium in which bids are a continuous, increasing function of signals where $b_i^1 = b_i^2 = \frac{s_i}{2}$ and $b_j^1 = b_j^2 = \frac{s_j}{2}$. As $b_i^1 = b_i^2\ \forall\ (s\ ,\ i)$ implicit bundling occurs.*

Proof. Assuming that it is never optimal to set $b_i^1 < b_i^2$ or $b_j^1 < b_j^2$ then Proposition 16.1 implies that the discriminatory auction decomposes into two separate single-unit, first-price auctions. When bidders have constant marginal valuations these auctions are symmetric. It is routine to verify that these strategies form a local optimum of the single-unit, first-price auction by substituting them into the first order conditions above. For example, if $b_j^2 = \frac{s_j}{2}$ then $\phi_j^2(b_i^1) = 2b_i^1$ and the first order condition for b_i^1 becomes $\frac{1}{2}s_i = b_i^1$. This is a global optimum as $\frac{\partial U_i}{\partial b_i^1} = \left[\frac{1}{2}\left[s_i + \phi_j(b_i)\right] - b_i\right] \phi\prime_j(b_i) - \phi_j(b_i) = s_i - 2b_i$ and therefore $\frac{\partial U_i}{\partial b_i^1} > 0$ if $b_i < \frac{1}{2}s_i$ and $\frac{\partial U_i}{\partial b_i^1} < 0$ if $b_i > \frac{1}{2}s_i$. By symmetry $b_j^2 = \frac{s_j}{2}$ is the equilibrium strategy for bidder j, and solving for b_i^2 and b_j^1 will give $b_i^1 = b_i^2 = \frac{s_i}{2}$ and $b_j^1 = b_j^2 = \frac{s_j}{2}$. This leads to implicit bundling as winning one unit when $b_i^1 > b_j^2$ implies that $b_i^2 > b_j^1$ also, so a bidders always wins both units or neither (as bids are continuous and increasing, the probability of ties is zero). Finally in these equilibria it is never optimal to set $b_i^1 < b_i^2$ or $b_j^1 < b_j^2$, consistent with the assumption made at the start of the proof. ■

One way to see that implicit bundling occurs is demonstrated in the Proposition 17.1. An intuitive way of considering the problem is to show that the discriminatory auction has some equivalence to the bundled auction when $b_i^1 = b_i^2 \; \forall \; (s \; , \; i)$ in equilibrium. As $b_j^1 = b_j^2$ the inverse bid functions are $\phi_j^1\left(b_i^2\right) = \phi_j^2\left(b_i^1\right)$, and because $b_i^1 = b_i^2$ the condition for bidder i to win one unit, $\phi_j^1\left(b_i^2\right) < s_j < \phi_j^2\left(b_i^1\right)$, becomes impossible. Graphically, the range of s_j over which it is possible to win one unit collapses to a point, so bidder i wins either two or zero units, as shown in Figure 17.1.

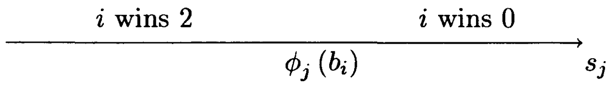


Figure 17.1: Implicit bundling

Therefore the discriminatory auction leads to implicit bundling, as in equilibrium one bidder always wins both units. Efficiency is not an issue when bidders have constant marginal valuations as any realisation is efficient. A revenue comparison can be made to the first-price auction for the bundle, in which it is an equilibrium to bid $b_i = \frac{1+a}{2}s_i = s_i$ when $\alpha = 1$, so the bidder with the highest signal wins both units and pays s_i. In the discriminatory auction the bidder with the highest signal wins both units and pays $b_i^1 + b_i^2 = \frac{s_i}{2} + \frac{s_i}{2} = s_i$. Therefore the discriminatory auction generates the same actual revenue as the first-price auction for the bundle if bidders have constant marginal valuations. The expected revenue raised will be $E\left[R\right] = E\left[Max\left[\frac{s_i}{2} + \frac{s_i}{2} \; , \; \frac{s_j}{2} + \frac{s_j}{2}\right]\right] = E\left[Max\left(s_i \; , \; s_j\right)\right] = \frac{2}{3}$.

17.2 The Vickrey Auction

For the Vickrey auction, Proposition 16.2 showed that the first order conditions for bids b_i^1 and b_j^2 are independent of b_j^1 and b_i^2, providing the requirement $b_i^1 \geq b_i^2$ does not constrain bids in equilibrium. When bidders have constant marginal valuations it will be shown that although $b_i^1 \geq b_j^2$ binds at the equilibrium bids, it does not constrain them, as in the discriminatory auction. This means that the auction can still be solved by assuming it decomposes into separate single-unit, second-price auctions in an equilibrium which also involves implicit bundling.

$$\begin{aligned} b_i^1 &: \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2\prime}\left(b_i^1\right) = 0 \\ b_j^2 &: \left[\frac{1}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right]\phi_i^{1\prime}\left(b_j^2\right) = 0 \end{aligned} \tag{17.1}$$

Proposition 17.2 *When bidders have constant marginal valuations, in equilibrium the Vickrey auction has a symmetric equilibrium in continuous, increasing strategies in which $b_i^1 = b_i^2 = s_i$ and $b_j^1 = b_j^2 = s_j$. As $b_i^1 = b_i^2\ \forall\ (s\ ,\ i)$ implicit bundling occurs.*

Proof. Assuming that it is never optimal to set $b_i^1 < b_i^2$ or $b_j^1 < b_j^2$ then Proposition 16.2 implies that the Vickrey auction decomposes into two separate single-unit, second-price, sealed-bid auctions. When bidders have constant marginal valuations these auctions are symmetric. It can be verified that the proposed strategies form an equilibrium of the single-unit auctions as if $b_j^2 = s_j$ then $\phi_j^2(b_i^1) = b_i^1$ and the first order condition for b_i^1 becomes $\frac{1}{2}\left[s_i - b_i^1\right] = 0$. This is a global optimum as $\frac{\partial U_i}{\partial b_i^1} = \frac{1}{2}\left[s_i - b_i^1\right]$ and therefore $\frac{\partial U_i}{\partial b_i^1} > 0$ if $b_i < s_i$ and $\frac{\partial U_i}{\partial b_i^1} < 0$ if $b_i > s_i$. By symmetry $b_j^2 = s_j$ is the equilibrium strategy for bidder j, and solving for b_i^2 and b_j^1 gives $b_i^1 = b_i^2 = s_i$ and $b_j^1 = b_j^2 = s_j$. This leads to implicit bundling as winning one unit when $b_i^1 > b_j^2$ implies that $b_i^2 > b_j^1$ also, so bidders always win both units or neither (as bids are continuous and increasing, the probability of ties is zero). Finally in these equilibria it is never optimal to set $b_i^1 < b_i^2$ or $b_j^1 < b_j^2$, consistent with the assumption made at the start of the proof. ∎

As with the discriminatory auction, the Vickrey auction leads to implicit bundling, as in equilibrium one bidder always wins both units. A revenue comparison can be made with the second-price auction for the bundle, in which it is an equilibrium to bid $b_i = (1 + a)\, s_i = 2s_i$ when $\alpha = 1$, so the bidder with the highest signal wins both units and pays the opponent's bid, $2s_j$. In the Vickrey auction the bidder with the highest signal wins both units and pays the sum of the opponent's bids $b_j^1 + b_j^2 = s_j + s_j = 2s_j$. Therefore the Vickrey auction generates the same actual revenue as the second-price auction for the bundle when bidders have constant marginal valuations. The expected revenue raised is $E[R] = E\left[Min\left[s_i + s_i\ ,\ s_j + s_j\right]\right] = 2E\left[Min\left(s_i\ ,\ s_j\right)\right] = \frac{2}{3}$.

The condition that $b_i^1 = b_i^2 = b_i = s_i$ follows from *fully symmetric* bidding strategies, in the sense that the separate second-price auctions into which the Vickrey auction decomposes are symmetric. In the case of constant marginal valuations these single-unit auctions also have equilibria of the form $b_i^1 = \frac{s_i + s_i^\theta}{2}$, $b_j^2 = \frac{s_j + s_j^{\frac{1}{\theta}}}{2}$, where $\theta > 0$, among others[1]. Although these are asymmetric equilibria of the single-unit auctions, they could form a symmetric equilibrium of the Vickrey auction where each bidder bids $\frac{s_i + s_i^\theta}{2}$ for a first unit and $\frac{s_i + s_i^{\frac{1}{\theta}}}{2}$ for a second unit. This is consistent with $b_i^1 \geq b_i^2 \ \forall$ $(s\ ,\ i)$ providing $0 < \theta \leq 1$, and some of these equilibrium generate low revenues. Any bid below $\frac{s_i}{2}$ for a second unit is weakly dominated, so a lower bound on the revenue in these equilibria is given by $E\left[R\right] = \frac{1}{2}$.

17.3 The Uniform Price Auction

When bidders have constant marginal valuations the uniform price auction in which bidders pay the highest losing bid has multiple equilibria in continuous strategies. Two extremes are the case of no *demand reduction*, where implicit bundling occurs, and *full demand reduction*, which generates no revenue. The first order conditions for the uniform price auction were derived in Equation 16.3.

$$\begin{aligned}
b_i^1 &: \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right] \phi_j^{2\prime}\left(b_i^1\right) = 0 \\
b_j^2 &: \left[\frac{1}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right] \phi_i^{1\prime}\left(b_j^2\right) = \phi_i^2\left(b_j^2\right) - \phi_i^1\left(b_j^2\right)
\end{aligned}$$

Proposition 17.3 *When bidders have constant marginal valuations the implicit bundling equilibrium of the Vickrey auction in which $b_i^1 = b_i^2 = s_i$ and $b_j^1 = b_j^2 = s_j$ is also an equilibrium of the uniform price auction.*

Proof. Proposition 17.2 demonstrates that there is a symmetric equilibrium of the Vickrey auction in which $b_i^1 = b_i^2 \ \forall \ (s\ ,\ i)$. Proposition 16.3 shows that an equilibrium of the Vickrey auction in which $b_i^1 = b_i^2 \ \forall \ (s\ ,\ i)$ is also an equilibrium of the uniform price auction. It is routine to verify Proposition 16.3 in this case by observing that if i bids $b_i^1 = b_i^2$ then $\phi_i^1\left(b_j^2\right) = \phi_i^2\left(b_j^1\right)$ so $\phi_i^2\left(b_j^2\right) - \phi_i^1\left(b_j^2\right) = 0$ and it follows that if this bidding strategy maximises expected surplus in the Vickrey auction then it also maximises expected surplus in the uniform price auction. ■

This equilibrium can be verified by substituting $b_j^1\left(s\right) = b_j^2\left(s\right) = s_j$ so that $\phi_j^1\left(b\right) = \phi_j^2\left(b\right) = b$ into the differentials of expected surplus with respect to b_i^1 and b_i^2 which give $\frac{\partial U_i}{\partial b_i^1} = \frac{1}{2}\left[s_i - b_i^1\right]$ and $\frac{\partial U_i}{\partial b_i^2} = \frac{1}{2}\left[s_i - b_i^2\right]$. Therefore $b_i^1 = b_i^2 = s_i$ constitutes a local maximum and by inspection also a global maximum. This equilibrium is equivalent to

[1] Milgrom (1981)

the Vickrey auction, as when $b_i^1 = b_i^2 \; \forall \; s_j$, bidders always win either two units or zero units (implicit bundling) and there is zero probability that their bid for a second unit can dictate the price they pay for the first. Therefore there is no *demand reduction* in this equilibrium.

The revenue raised in this equilibrium of the uniform price auction is the same as in the Vickrey auction derived in Proposition 17.2. In the Vickrey auction the bidder with the highest signal wins both units and pays the sum of the opponent's bids $b_j^1 + b_j^2 = s_j + s_j = 2s_j$. In the uniform price auction the bidder with the highest signal wins both units and pays double the highest losing bid $2b_j^1 = 2s_j$. Therefore this equilibrium of the uniform price auction generates the same actual revenue as the Vickrey auction and hence the second-price auction for the bundle. The expected revenue raised is $E[R] = E[Min(2s_i \, , \, 2s_j)] = \frac{2}{3}$.

Proposition 17.4 *When bidders have constant marginal valuations it is a Nash equilibrium of the highest losing bid uniform price auction for bidders to bid* $b_i^1 = \frac{s_i+1}{2}$ *and* $b_j^1 = \frac{s_j+1}{2}$ *for first units and* $b_i^2 = b_j^2 = 0$ *for second units.*

Proof. Assume that $b_j^1(s_j) = \frac{s_j+1}{2}$ and $b_j^2(s_j) = 0$. As $\frac{s_i+s_j}{2} - b_j^2 = \frac{s_i+s_j}{2} - 0 > 0$ $\forall \; s_i \, , \, s_j$, any bid b_i^1 for i which always wins the first unit is optimal. If bidder i bids 0 for the second unit, both bidders receive a payoff of $\frac{s_i+s_j}{2}$. Any bid b_i^2 that wins an additional unit always reduces bidder i's payoff as $2\left(\frac{s_i+s_j}{2} - b_j^1\right) = s_i - 1 < \frac{s_i+s_j}{2}$. Any positive bid b_i^2 that does not win an additional unit is weakly dominated by $b_i^2 = 0$ as increasing b_i^2 above 0 only increases the amount paid on the first unit, directly reducing surplus. Therefore $b_i^2 = 0$ weakly dominates any $b_i^2 = 0$ when the opponent bids $b_j^1(s_j) = \frac{s_j+1}{2}$ for a first unit. ■

This Nash equilibrium of the uniform price auction generates no revenue, as bidders engage in *full demand reduction*, submitting zero bids for their second units so the highest losing bid is zero. An explanation that the revenue equivalence theorem does not hold is linked to the interim allocation and payments, which differ from the implicit bundling which arises in the other auctions considered in this section. As this equilibrium maximises expected surplus to bidders, it may be focal, making the uniform price auction undesirable. This may be especially important when bidders are boundedly rational, as it corresponds to the simple analogy of an *equal division* of the units. This will be discussed in Chapter Twenty-One.

17.4 Discussion

In the case of constant marginal valuations any allocation is efficient. The allocations arising in equilibrium always lead to the bidder with the highest signal receiving both units, with one exception: in the *full demand reduction* equilibrium of the uniform price auction, both bidders submit bids of zero for their second units and so each receives one unit regardless of their signals. This is equivalent to the low revenue equilibria discussed in Engelbrecht-Wiggans and Khan (1998) for the case when bidders have independent private values. In the other auctions bidders submit flat demand curves which are not supported empirically. Chapter 18 will show that downward sloping bid curves will arise when bidders have decreasing marginal valuations.

All of the auctions considered above have a symmetric equilibrium in which revenue equivalence holds, and these generate the same expected revenue as the auction for the bundle, $E[R] = E\left[2\max\left(\frac{s_i}{2}\right)\right] = \frac{2}{3}$. The discriminatory price auction generates the same actual revenue as the first-price, sealed-bid auction for the bundle. The Vickrey auction and the equilibrium of the uniform price auction without *demand reduction* generate the same actual revenue as the second-price, sealed-bid auction for the bundle. However, it was shown that the Vickrey auction also has additional equilibria, both symmetric and asymmetric, which generate low revenues. The uniform price auction has a *full demand reduction* equilibrium in which bidders bid aggressively for a first unit and submit zero bids for a second unit, leading to zero revenue for the auctioneer.

These insights can be summarised in the following claims:

Claim 17.1 ***Efficiency:*** *When $\alpha = 1$ bidders have constant marginal valuations and as any allocation is efficient, any auction allocating both units to bidders is efficient.*

Claim 17.2 ***Expected Revenue:*** *When $\alpha = 1$ bidders have constant marginal valuations and the auctions can be ranked by the expected revenue they provide to the seller as follows:*

$$\begin{aligned} ER\,[\textit{first-price bundle}] &= ER\,[\textit{Discriminatory}] \geq \\ ER\,[\textit{second-price bundle}] &= ER\,[\textit{Vickrey}] \geq \\ & \quad ER\,[\textit{Uniform}] \end{aligned}$$

The first inequality binds when the equilibrium of the Vickrey or second-price auction is fully symmetric. The second inequality binds in the equilibrium of the uniform price auction in which there is no demand reduction.

Chapter 18

Decreasing Marginal Valuations

The case of decreasing marginal valuations could be motivated by diminishing marginal utility for the object being auctioned.[1] The first unit has a valuation of $\frac{s_i+s_j}{2}$ and the second unit is a constant proportion $\frac{1}{2} < \alpha < 1$ of this, so the gross utility from winning both units is $(1+\alpha)\,\frac{s_i+s_j}{2}$. Although bidders are symmetric, introducing diminishing marginal valuations creates asymmetry. To win a unit bidder i must compete against bidder j's bid on a second unit, for which j has a lower marginal valuation. When bidders have constant and increasing marginal valuations, the discriminatory and Vickrey auctions lead to implicit bundling because $b_i^1 = b_i^2\ \forall\ (s\ ,\ i)$ in equilibrium. This chapter will show that when marginal valuations are decreasing, these auctions decompose into multiple asymmetric single-unit auctions.

18.1 The Discriminatory Auction

Proposition 16.1 demonstrated that providing $b_i^1 \geq b_i^2\ \forall\ (s\ ,\ i)$ does not constrain the optimal bidding strategies in equilibrium, the discriminatory auction decomposes into two separate first-price, sealed-bid auctions for single units, where bidder i competes for a first unit against bidder j's bid for a second unit and vice versa. This can be observed as the first order conditions for bids b_i^1 and b_j^2 shown in Equations 18.1 and 18.2, which were derived in Proposition 16.1, are independent of b_j^1 and b_i^2. This section will solve these first order conditions to find the equilibrium bidding strategies in the asymmetric single-unit auctions, and show that these also form a symmetric

[1]Even if the payoffs are constant, these could be motivated implicitly as the reduced form of risk aversion, or even reciprocity considerations.

equilibrium of the discriminatory auction.

$$b_i^1 \ : \ \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2'}\left(b_i^1\right) = \phi_j^2\left(b_i^1\right) \tag{18.1}$$

$$b_j^2 \ : \ \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_2^2\right]\phi_i^{1'}\left(b_j^2\right) = \phi_i^1\left(b_j^2\right) \tag{18.2}$$

Proposition 18.1 *When* $\frac{1}{2} < \alpha < 1$ *bidders have diminishing marginal valuations and the discriminatory auction has a symmetric Bayesian Nash equilibrium in which it decomposes into two asymmetric single-unit, first-price, sealed-bid auctions. In equilibrium bids satisfy the closed form Equations* 18.3 *and* 18.4.

Proof. Assume that $b_i^1 \geq b_i^2 \ \forall \ (s\ ,\ i)$ does not constrain the optimal bids in equilibrium, so following Proposition 16.1 the discriminatory auction decomposes into multiple single-unit, first-price, sealed-bid auctions. The auction for bidder i's first unit and bidder j's second unit is solved as a single-unit asymmetric auction in Appendix C.1, giving a closed form solution shown in Equations 18.3 and 18.4. It is straightforward but tedious to verify that bids are global optima[2] and that the condition that $b_i^1 \geq b_i^2 \ \forall \ (s\ ,\ i)$ does not constrain the optimum bids, which is illustrated in Figure 18.1. ∎

Although the overall setting is symmetric, the discriminatory auction decomposes into asymmetric single-unit auctions, because bidder i competes for a first unit against bidder j's second unit, so it is worth $\frac{s_i+s_j}{2}$ to bidder i but only $\alpha\frac{s_i+s_j}{2}$ to bidder j. The asymmetry affects bids directly because of the difference in player's valuations. It also has an indirect effect since compared to a symmetric auction, the winner's curse is amplified for bidder j, as winning a second unit conveys especially bad news about s_i, bidder i's signal. For bidder i the winner's curse is mitigated, as winning a first unit is not as bad news about s_j, as the unit is worth less to bidder j.

The equilibrium bidding strategies for bidder i are derived in Appendix C.1 and are given in closed form in Equations 18.3 and 18.4.

$$b_i^1 \ : \ \frac{\left[-\alpha + \sqrt{\frac{\alpha(1-\alpha)}{b_i^1}s_i + \alpha^2}\right]^{\alpha}}{\left[2 - \alpha - \sqrt{\frac{\alpha(1-\alpha)}{b_i^1}s_i + \alpha^2}\right]^{(2-\alpha)}} = \left[b_i^1 s_i\left(\frac{1+\alpha}{\alpha\left(1-\alpha\right)^2}\right)\right]^{(1-\alpha)} \tag{18.3}$$

$$b_i^2 \ : \ \frac{\left[1 - 2\alpha + \sqrt{1 - s_i\frac{\alpha(1-\alpha)}{b_i^2}}\right]^{(2\alpha-1)}}{1 - \sqrt{1 - s_i\frac{(1-\alpha)\alpha}{b_i^2}}} = \left[b_i^2 s_i\left(\frac{1+\alpha}{\alpha\left(1-\alpha\right)^2}\right)\right]^{(1-\alpha)} \tag{18.4}$$

[2]It is possible to show that $\frac{\partial U_i}{\partial b_i^k}$ is positive if $b_i^k < b_i^{k*}$ and negative if $b_i^k > b_i^{k*}$ $\forall k, i$ where b_i^{k*} satisfies the first order conditions below.

Identical conditions apply for b_j^1 and b_j^2 by symmetry.

Figure 18.1 illustrates the equilibrium bids for $\alpha = \frac{3}{5}$. The upper line shows b_i^1, bidder i's equilibrium bid for a first unit as a function of the signal s_i. The lower line shows b_i^2, bidder i's equilibrium bid for a second unit. The support of both the bids is $\left[0\,,\ \frac{\alpha}{1+\alpha}\right]$. It is necessary for them to be the same, as in the equilibrium of each single-unit auction, bidder i's maximum bid for a first unit $b_i^1(1)$ must equal bidder j's maximum bid for a second unit, $b_j^2(1)$, otherwise one of the bidders could reduce their bid and still win with certainty. Therefore $b_i^1(1) = b_j^2\,(1)$ and $b_j^2\,(1) = b_i^2\,(1)$ by symmetry.

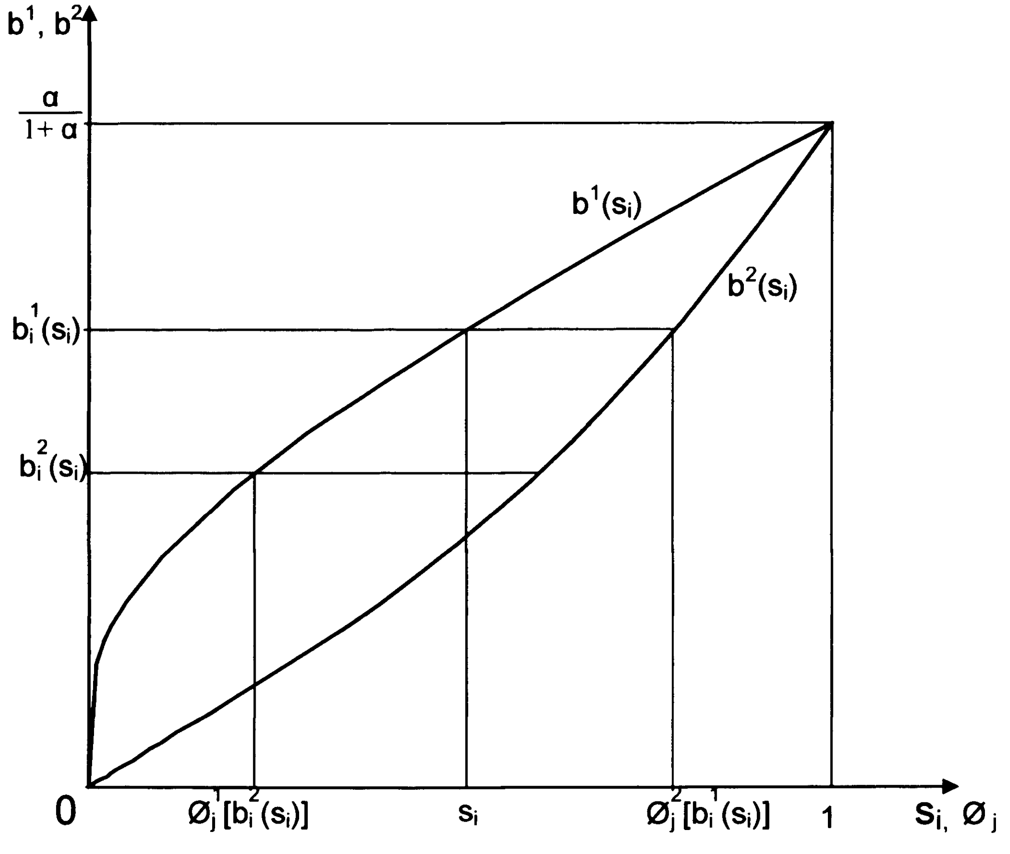


Figure 18.1: Equilibrium bids in the discriminatory auction with diminishing marginal valuations when $\alpha = \frac{3}{5}$

As $\alpha \longrightarrow 1$ the equilibrium bids tend towards $b_i(s_i) = b_i^1(s_i) = b_i^2(s_i) = \frac{s_i}{2}$, the equilibrium of the discriminatory auction when bidders have constant marginal valuations. This is illustrated in Figure 18.2. For each value of α the upper line shows the equilibrium bid for the first unit, while the lower line shows the equilibrium bid for a second unit.

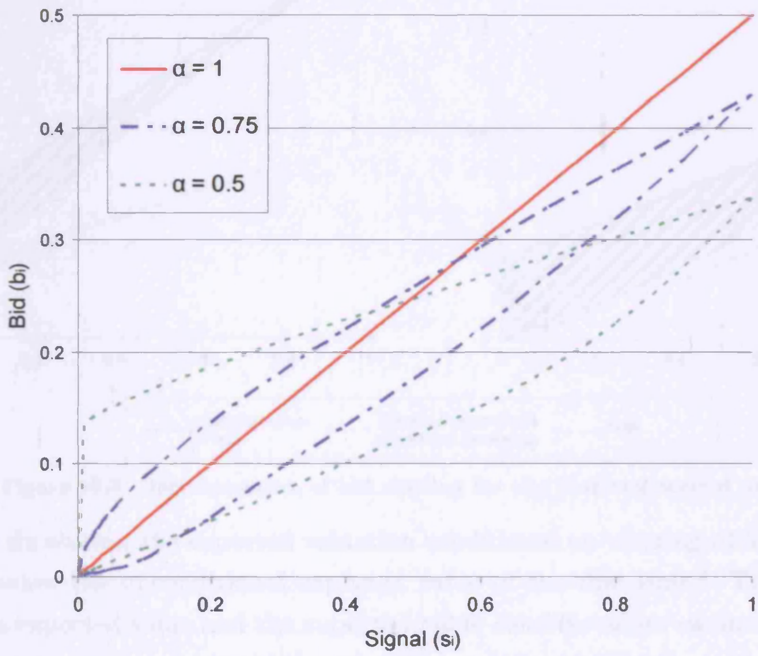


Figure 18.2: Equilibrium bids in the discriminatory auction with diminishing marginal valuations for different α

In common value first-price auctions, bidders reduce their bids below the expected value of a unit for two reasons, firstly to avoid the winner's curse and secondly to extract positive surplus. These concepts are interdependent in equilibrium as both affect the opponent's optimal strategy, which in turn affects a bidder's own strategy. In a Bayesian Nash equilibrium these effects are incorporated in bidder's strategies, and the extent to which bids are reduced below expected values can be decomposed into two effects: surplus extraction and the winner's curse. This is illustrated in Figure 18.3 for the case when $\alpha = \frac{3}{5}$.

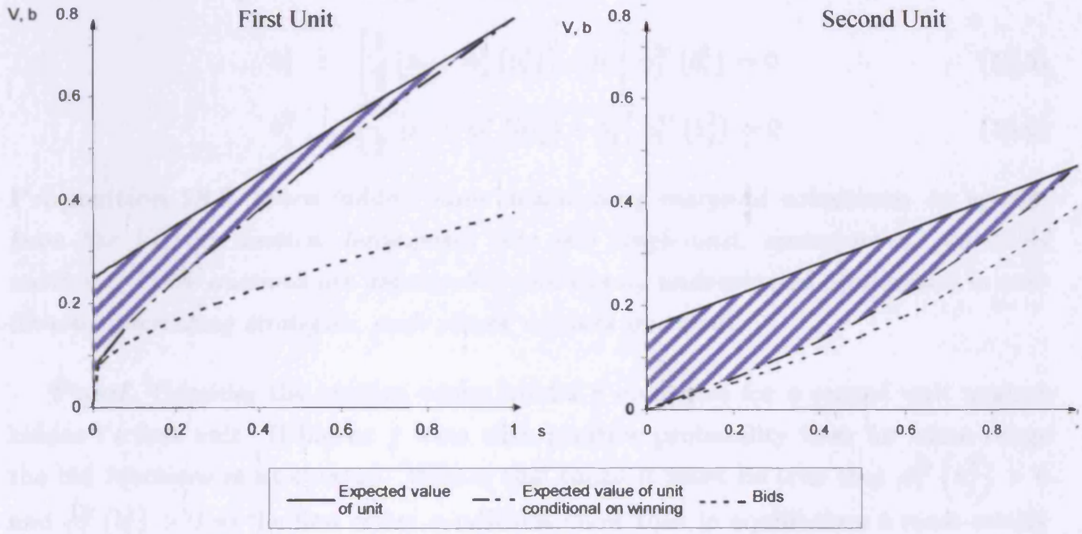


Figure 18.3: Decomposition of bid shading for the first and second units

Figure 18.3 shows that the expected valuation conditional on winning *at least* one unit is strictly below the unconditional expected value of the first unit.[3] The difference between the expected value and the expected value *conditional on winning* represents the adjustment necessary to avoid the winner's curse in equilibrium, and is shown by the shaded areas in this figure for both the first and second units. At very low signals winning any units is bad news, so bidders avoid the winner's curse by substantially reducing their bids on both units. If a bidder has a very high signal the winner's curse has little effect, but bidders still reduce their bids to extract surplus. Figure 18.3 shows that the greatest asymmetry in the impact of the winner's curse arises when bidders have moderate signals, as it does not greatly affect bidding for a first unit but is substantial when bidding on a second unit. Chapter Twenty-One will reexamine this result, and discuss the experimental evidence that inexperienced bidders may fail to adjust their expectations fully to account for the winner's curse.

18.2 The Vickrey Auction

Proposition 16.2 showed that the Vickrey auction could decompose into two asymmetric single-unit auctions, as in equilibrium the first order conditions for bids b_i^1 and b_j^2

[3] Winning *only* one unit is good news about value if a bidder has a high signal, so conditioning on this will raise the expected valuation. Although the model has been developed using the probabilities and expectations conditional on winning only one or only two units, conditioning on winning at least one unit and a second unit conditional on winning the first gives identical results.

shown in Equations 18.5 and 18.6 are independent of b_j^1 and b_i^2, providing $b_i^1 \geq b_i^2$ does not constrain the optimal bidding strategies.

$$b_i^1 \ : \ \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right] \phi_j^{2\prime}\left(b_i^1\right) = 0 \tag{18.5}$$

$$b_j^2 \ : \ \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right] \phi_i^{1\prime}\left(b_j^2\right) = 0 \tag{18.6}$$

Proposition 18.2 *When bidders have diminishing marginal valuations, in equilibrium the Vickrey auction decomposes into two single-unit, second-price, sealed-bid auctions. These auctions are asymmetric and in any undominated equilibrium in continuous, increasing strategies, each player receives one unit.*

Proof. Consider the auction where bidder j competes for a second unit against bidder i's first unit. If bidder j wins with positive probability then for some range the bid functions must overlap. Within this range it must be true that $\phi_i^{1\prime}\left(b_j^2\right) > 0$ and $\phi_j^{2\prime}\left(b_i^1\right) > 0$ so the first order conditions show that in equilibrium b must satisfy $b_i^1 = \frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right]$ and $b_j^2 = \frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right]$. However, it is impossible for $b = \frac{1}{2}\left[\phi_i^1(b) + \phi_j^2(b)\right] = \frac{\alpha}{2}\left[\phi_i^1(b) + \phi_j^2(b)\right]$ as $\alpha < 1$. Therefore the bids cannot overlap and bidder i must always win, so that $b_i^1 \geq b_i^2 \ \forall \ (s\ ,\ i)$. In a symmetric equilibrium $b_i^2(\tilde{s}) = b_j^2(\tilde{s}) \ \forall \ \tilde{s}$ and $b_i^1(\tilde{s}) = b_j^1(\tilde{s}) \ \forall \ \tilde{s}$ so $b_i^1 > b_j^2 \ \forall \ s_i\ ,\ s_j$ implies $b_i^1 > b_i^2 \ \forall \ s$ and $b_j^1 > b_j^2 \ \forall \ s$ which satisfies the requirements for the Vickrey auction to decompose into two single asymmetric auctions in equilibrium. ■

In a second-price auction the direct effect of the asymmetry when $\alpha < 1$ is to reduce the winner's curse for a player bidding for a first unit, and increase it for a player who is bidding on a second. This means that players can bid more aggressively for the first unit but must bid more conservatively on the second. However, given that the opponent is not only advantaged, but is also expected to bid aggressively, the winner's curse becomes even more severe when bidding on a second unit, and so on. This indirect effect is powerful and means that each player wins one unit in equilibrium even if α is close to 1.[4] In addition, these equilibria tend to generate low revenues.

Proposition 18.3 *When players have diminishing marginal valuations it is an ex post equilibrium of the Vickrey auction for players to bid $b_i^1 = \frac{s_i+1}{2}$ for their first unit and $b_i^2 = \alpha\frac{s_i}{2}$ for a second unit.*

Proof. Consider the auction where bidder j competes for a second unit against bidder i's first unit. As $b_i^1 = \frac{s_i+1}{2} > \alpha\frac{s_j}{2} = b_j^2 \ \forall \ s_i\ ,\ s_j$ bidder i always wins. As

[4] Klemperer (1998) shows that this effect can have a dramatic effect on the equilibrium in ascending auctions, where one player has a small additive advantage.

$\frac{s_i+s_j}{2} - b_i^2 = \frac{s_i+s_j}{2} - \alpha\frac{s_i}{2} > 0 \; \forall \; s_i \; , \; s_j$, any bid for i which always wins is optimal. Likewise as $\alpha\frac{s_i+s_j}{2} - b_i^2 = \alpha\frac{s_i+s_j}{2} - \frac{s_i+1}{2} < 0 \; \forall \; s_i \; , \; s_j$, any bid which always loses for bidder j is optimal. It should be noted that any bid above $\frac{s_i+1}{2}$ on a first unit or below $\alpha\frac{s_i}{2}$ on a second unit is weakly dominated, so these strategies put a lower bound on revenue for the auctioneer. The multi-unit auction decomposes as $\frac{s_i+1}{2} > \alpha\frac{s_i}{2}$ and $\frac{s_j+1}{2} > \alpha\frac{s_j}{2}$ imply that $b_i^2 > b_i^1 \; \forall \; s \; , \; i$. ■

When players have diminishing marginal valuations then any equilibrium of the Vickrey auction leads to an efficient allocation, with each bidder receiving one unit, but expected revenue is low. A lower bound on revenue is given by the equilibrium in which $b_i^1 = \frac{s_i+1}{2}$ and $b_i^2 = \alpha\frac{s_i}{2}$, where $E[R] = E\left[\frac{\alpha}{2}s_1 + \frac{\alpha}{2}s_2\right] = \frac{\alpha}{2}$, while an upper bound occurs when $b_i^1 = \frac{s_i+1}{2}$ and $b_i^2 = \frac{s_i}{2}$, where $E[R] = E\left[\frac{s_1}{2} + \frac{s_2}{2}\right] = \frac{1}{2}$.

18.3 The Uniform Price Auction

The first order conditions of the uniform price auction, where bidders pay the highest losing bid, were derived in Proposition 16.3 and are shown in Equations 18.7 and 18.8. When bidders have diminishing marginal valuations and strategies are continuous, it is no longer an equilibrium for players to bid the same for their second unit as their first, as the asymmetries arising in the Vickrey auction force the equilibrium of the uniform price auction to *full demand reduction*, where bidders submit zero bids for their second units and the auctioneer receives no revenue.

$$\frac{\partial U_i}{\partial b_i^1} = \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2\prime}\left(b_i^1\right) \qquad (18.7)$$

$$\frac{\partial U_j}{\partial b_j^2} = \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right]\phi_i^{1\prime}\left(b_j^2\right) + \phi_i^1\left(b_j^2\right) - \phi_i^2\left(b_j^2\right) \qquad (18.8)$$

Proposition 18.4 *When bidders have diminishing marginal valuations and use continuous, increasing strategies, then in the equilibrium of the highest losing bid uniform price action each bidder receives one unit.*

Proof. If bidder j wins a second unit with positive probability then for some range the bid functions b_i^1 and b_j^2 must overlap. Within this range it must be true that $\phi_j^{2\prime}\left(b_i^1\right) > 0$ and $\phi_i^{1\prime}\left(b_j^2\right) > 0$ so the first order conditions show that in equilibrium b must satisfy $b_i^1 = \frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right]$ and $b_j^2 \leq \frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right]$. The inequality arises due to a *demand reduction* effect, as $b_j^1 \geq b_j^2$ implies that $\phi_i^1\left(b_j^2\right) - \phi_i^2\left(b_j^2\right) \leq 0$ and therefore $\frac{\partial U_j}{\partial b_j^2} \leq 0$ even when $b_j^2 = \frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right]$. Even in the case of no *demand* *reduction*, it is impossible for $\frac{1}{2}\left[\phi_i^1(b)+\phi_j^2(b)\right] = b \leq \frac{\alpha}{2}\left[\phi_i^1(b)+\phi_j^2(b)\right]$ as $\alpha < 1$. Therefore the bids cannot overlap and bidder i must always win, so that $b_i^1 > b_j^2 \ \forall \ s_i \ , \ s_j$. In a symmetric equilibrium $b_i^2(s) = b_j^2(s)$ and $b_i^1(s) = b_j^1(s)$ so $b_i^1 > b_j^2 \ \forall \ s_i \ , \ s_j$ implies $b_i^1 > b_i^2 \ \forall \ s_i \ , \ s_j$ and $b_j^1 > b_j^2 \ \forall \ s_i \ , \ s_j$ which means that $b_i^1 \geq b_i^2 \ \forall \ (s \ , \ i)$ never constrains the equilibrium. ■

An intuitive way to consider the uniform price auction is to take the limiting case, where there is no *demand reduction*. This is equivalent to the Vickrey auction derived above, and decomposes into two separate asymmetric second-price auctions. Secondly, observe that with *demand reduction*, $b_j^2 < \frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right]$, so the requirement that $\frac{1}{2}\left[\phi_i^1(b)+\phi_j^2(b)\right] = b < \frac{\alpha}{2}\left[\phi_i^1(b)+\phi_j^2(b)\right]$ is impossible even if $\alpha = 1$. In the second-price Vickrey auction, asymmetry alone is sufficient to generate an equilibrium in which each bidder wins one unit because of the way it interacts with the winner's curse. If in addition bidders reduce their bids on the second unit because of *demand reduction*, this reinforces the asymmetry as winning with a reduced bid is even worse news.

Proposition 18.5 *When players have diminishing marginal valuations it is an ex post equilibrium of the highest losing bid uniform price auction for players to bid* $b_i^1 = \frac{s_i+1}{2}$ *for their first units and* $b_i^2 = 0$ *for their second units. This generates zero revenue for the auctioneer.*

Proof. Consider the case when bidder j competes for a second unit against bidder i's first unit. As $b_i^1 = \frac{s_i+1}{2} > 0 = b_j^2 \ \forall \ s_i \ , \ s_j$ bidder i always wins. As $\frac{s_i+s_j}{2} - b_j^2 = \frac{s_i+s_j}{2} - 0 > 0 \ \forall \ s_i \ , \ s_j$, any bid for i which always wins the first unit is optimal. Likewise as $\alpha\frac{s_i+s_j}{2} - b_i^2 = \alpha\frac{s_i+s_j}{2} - \frac{s_i+1}{2} < 0 \ \forall \ s_i \ , \ s_j$, it is optimal for bidder j to bid sufficiently low to always lose the second unit. Given that each bidder always wins one unit in equilibrium, the payoff to each bidder equals $\frac{s_i+s_j}{2} - Max\left(b_i^2 \ , \ b_j^2\right)$. Any non-zero bid for a second unit is weakly dominated by a strategy of bidding $b_i^2 = b_j^2 = 0$, so in equilibrium each bidder obtains a net payoff of $\frac{s_i+s_j}{2}$ and the auctioneer receives no revenue. In this case it is straightforward to confirm that $b_i^2 > b_i^1 \ \forall \ s_i$ as $\frac{s_i+1}{2} > 0$. ■

The almost common value problem in the Vickrey auction[5] extends to the uniform price auction when bidders pay the highest losing bid. In addition, as they never win a second unit in equilibrium, it is a weakly dominant strategy for bidders to submit zero bids for a second unit. The effect of asymmetry forces the equilibrium to that of *full demand reduction*. The outcome of the highest losing bid uniform price auction is efficient, but generates zero revenue for the auctioneer.

[5] First discussed in Bihkchandani (1988).

18.4 Discussion

As bidders have diminishing marginal valuations the efficient outcome is to allocate one unit to each bidder. The uniform price and Vickrey auctions always allocate the units efficiently, while the bundled auction always allocates them inefficiently. The discriminatory auction is generally inefficient, although it is more likely to achieve an efficient outcome the lower the value of α. The equilibrium allocations in the discriminatory auction when $\alpha = \frac{3}{5}$ are illustrated in Figure 18.4 for different values of s_i and s_j. The shaded region shows the efficient allocation (1 , 1) which occurs in equilibrium when bidders have similar, moderate signals.

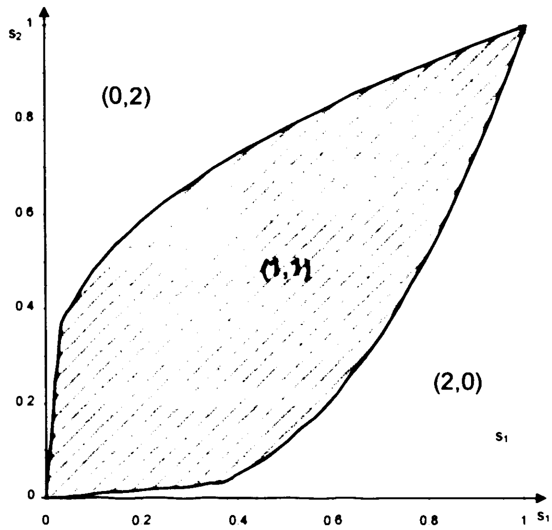


Figure 18.4: Equilibrium allocation of units in the discriminatory auction

As the different auctions have different interim allocation rules, the expected revenue theorem does not hold. To motivate the revenue rankings, the expected total payment when $\alpha = \frac{3}{5}$ made by a bidder with a signal s_i is illustrated in Figure 18.5.

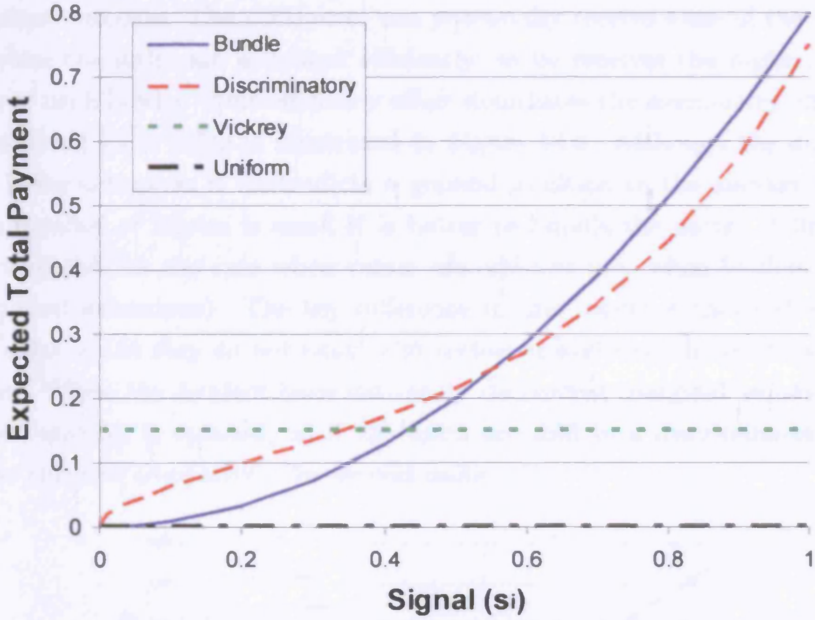


Figure 18.5: Expected payment for a bidder with signal s_i when $\alpha=\frac{3}{5}$

Figure 18.5 illustrates two important points. Firstly, bidders with low signals expect to pay more in a discriminatory auction than in a first-price auction for the bundle because they are more likely to win a unit, while bidders with high signals expect to pay more in the bundled auction because they are more likely to win two units. Secondly this is reinforced by the effect of asymmetries on the winner's curse. A bidder with a low signal bids more in total in the discriminatory price auction than in the auction for the bundle, so $b_i^1 + b_i^2 > b_i$ (b_i is the bid for the bundle). A bidder with a high signal bids more in the auction for the bundle than the total bids in the discriminatory auction. Both the discriminatory and bundled auctions generate more revenue than the Vickrey and uniform price auctions. The second-price auctions, although they are efficient, generate low revenues as implicit asymmetries arise when the multi-unit auctions decompose into single-unit auctions. Even if α is close to 1, they create an almost common values problem in which the impact of the winner's curse is severe.

Focusing on the comparison of the auction for the bundle with the discriminatory auction, there are two fundamental influences on expected revenue. Firstly the asymmetry between bidders reduces the competitiveness in the discriminatory auction,[6]

[6]This is because of the winner's curse.

reducing expected revenue compared to the symmetry of bundling. Secondly, bidders have higher valuations for their first units and the discriminatory auction may lead to an efficient outcome. The auctioneer can potentially receive some of the additional surplus when the units are allocated efficiently, as he receives the higher, first unit payments of each bidder. This efficiency effect dominates the asymmetry effect in the cases considered ($\alpha > 0.5$), as illustrated in Figure 18.6. Although the difference is slight, it is important as it contradicts a general intuition in the auction literature: when the number of buyers is small it is better to bundle the units. Palfrey (1983) demonstrated this for the case when values are additive (i.e. when bidders have constant marginal valuations). The key difference in this model is that bidders have a *common value which they do not know with certainty,* and have diminishing marginal valuations. When the bidders have extremely decreasing marginal valuations, it is clear that bundling is optimal, as if the units are sold in a discriminatory auction there is no effective competition for second units.

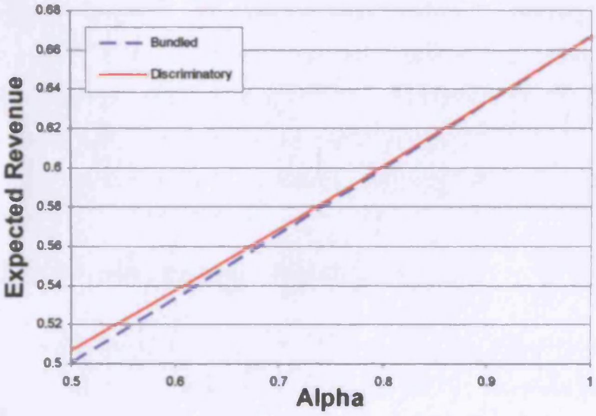


Figure 18.6: Expected Revenue of Discriminatory and First-Price Bundle Auction for Different α

To summarise, Table 18.1 evaluates the expected revenues of all the auctions considered when $\alpha = \frac{3}{5}$.

Auction	Discriminatory	Vickrey	Uniform	First-Price For Bundle
Expected Revenue	0.537	0.3	0	0.533

Table 18.1

Claim 18.1 *Efficiency: When $\frac{1}{2} < \alpha < 1$ so bidders have diminishing marginal valuations, the Vickrey and Uniform price auctions are efficient, allocating one unit to each* *bidder. The discriminatory auction is not efficient although it can allocate the units efficiently when both bidders receive similar, moderate signals. The bundled auction never allocates the units efficiently.*

Claim 18.2 *Expected Revenue: When $\frac{1}{2} < \alpha < 1$ the auctions can be ranked by the expected revenue they provide to the seller as follows:*

$$\begin{aligned} ER\,[\textit{Discriminatory}] &> \\ ER\,[\textit{First-Price Bundle}] &\geq ER\,[\textit{second-price bundle}] \\ &> ER\,[\textit{Vickrey}] \\ &> ER\,[\textit{Uniform}] \end{aligned}$$

The inequality binds when the equilibrium of the second-price auction is symmetric.

Chapter 19

Increasing Marginal Valuations

Increasing marginal valuations lead to implicit bundling in a discriminatory auction (and in a Vickrey auction). It was assumed that bidders are constrained to submitting downward sloping bids so that $b_i^1 \geq b_i^2 \ \forall \ (s \ , \ i)$, which are aggregated into a demand curve by the auctioneer. Even in this case, it will be demonstrated that in equilibrium the discriminatory and Vickrey auction lead to implicit bundling as $b_i^1 = b_i^2$ always binds. The case of combinatorial bidding will be discussed at the end of this section, although for reasonable allocation rules in equilibrium $b^2 \geq b^1 \ \forall \ s$ for both bidders, which is sufficient to generate implicit bundling anyway. It is assumed that $1 < \alpha < 2$.

19.1 The Discriminatory Auction

When bidders are constrained to submit decreasing bid schedules, it was shown in Proposition 17.1 that the constraint $b_i^1 \geq b_i^2$ binds in the equilibrium of the discriminatory auction with constant marginal valuations, leading to implicit bundling. This section will argue that this is also a constrained equilibrium of the discriminatory auction when bidders have increasing marginal valuations. In Proposition 16.1 the following first order conditions for the discriminatory auction were derived:

$$\frac{\partial U_1}{\partial b_i^1} = \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right]\phi_j^{2\prime}\left(b_i^1\right) - \phi_j^2\left(b_i^1\right) \tag{19.1}$$

$$\frac{\partial U_1}{\partial b_i^2} = \left[\frac{\alpha}{2}\left[s_i + \phi_j^1\left(b_i^2\right)\right] - b_i^2\right]\phi_j^{1\prime}\left(b_i^2\right) - \phi_j^1\left(b_i^2\right) \tag{19.2}$$

Proposition 19.1 *When bidders have increasing marginal valuations the discriminatory auction has a symmetric equilibrium in which bids are a continuous, increasing function of signals where* $b_i^1 = b_i^2 = \frac{1+\alpha}{4}s_i$ *and* $b_j^1 = b_j^2 = \frac{1+\alpha}{4}s_j$. *As* $b_i^1 = b_i^2 \ \forall \ (s \ , \ i)$ *implicit bundling occurs.*

Proof. Assuming the opponent bids $b_j^1 = b_j^2 = \frac{1+\alpha}{4} s_j$ and that the constraint $b_i^1 = b_i^2$ binds, then bidder i's optimisation problem reduces to that of the first-price auction for the bundle derived in Proposition 16.4, where it was shown that $b_i = b_i^1 + b_i^2 = \frac{1+\alpha}{2} s_i$ is a global optimum, therefore in this case $b_i^1 = b_i^2 = \frac{1+\alpha}{4} s_i$. It is necessary to show that for this to be an optimum $\frac{\partial U_i}{\partial b_i^1} \leq 0$ and $\frac{\partial U_i}{\partial b_i^2} \geq 0$ for all signals, so that $b_i^1 \geq b_i^2$ always binds. Substituting the equilibrium conditions into Equations 19.1 and 19.2 it is possible to show that $\frac{\partial U_i}{\partial b_i^1} = 2s_i \left(\frac{1-\alpha}{1+\alpha}\right) \leq 0 \ \forall \ s_i$ and $\frac{\partial U_i}{\partial b_i^2} = 2s_i \left(\frac{2\alpha-1}{1+\alpha}\right) \geq 0 \ \forall \ s_i$. By symmetry the same is true for bidder j and the strategies constitute an equilibrium where implicit bundling occurs as winning one unit when $b_i^1 > b_j^2$ implies that $b_i^2 > b_j^1$ also, so bidders always win both units or neither (as bids are continuous and increasing, the probability of ties is zero). ■

Therefore the discriminatory case leads to implicit bundling, as in equilibrium one bidder always wins both units. As bidders have increasing marginal valuations this is always efficient and the revenue raised is the same as that for the first-price auction for the bundle.

19.2 The Vickrey Auction

In Chapter Sixteen, Proposition 16.2 demonstrated that the Vickrey auction generated the following first order conditions:

$$b_i^1 \ : \ \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right] \phi_j^{2\prime}\left(b_i^1\right) = 0 \tag{19.3}$$

$$b_j^2 \ : \ \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right] \phi_i^{1\prime}\left(b_j^2\right) = 0 \tag{19.4}$$

Proposition 19.2 *When bidders have increasing marginal valuations, in equilibrium the Vickrey auction has a symmetric equilibrium in continuous, increasing strategies in which* $b_i^1 = b_i^2 = \frac{1+\alpha}{2} s_i$ *and* $b_j^1 = b_j^2 = \frac{1+\alpha}{2} s$. *As* $b_i^1 = b_i^2 \ \forall \ (s \ , \ i)$ *implicit bundling occurs.*

Proof. Assuming the opponent bids $b_j^1 = b_j^2 = \frac{1+\alpha}{2} s_j$ and that the constraint $b_i^1 \geq b_i^2$ binds, then bidder i's optimisation problem reduces to that of the second-price auction for the bundle in which $b_i = b_i^1 + b_i^2 = (1+\alpha) s_i$ is a Bayesian Nash equilibrium,[1] therefore in this case $b_i^1 = b_i^2 = \frac{1+\alpha}{2} s_i$. It is necessary to show that for this to be an optimum $\frac{\partial U_i}{\partial b_i^1} \leq 0$ and $\frac{\partial U_i}{\partial b_i^2} \geq 0$ for all signals, so that $b_i^1 \geq b_i^2$ always

[1] For bidder i, differentiating U_i gives $\frac{\partial U_i}{\partial b_i} = \left[\frac{1+\alpha}{2}\left[s_i + \phi_j\left(b_i\right)\right] - b_i\right] \phi_j'\left(b_i\right) = 0$. Assuming $b_j = (1+\alpha) s_j$ so $\phi_j(b) = \frac{1}{1+\alpha} b$ then $\frac{\partial U_i}{\partial b_i} = \frac{1}{2}\left[s_i - \frac{b_i}{1+\alpha}\right]$ which is optimised at $b_i = (1+\alpha) s_i$. This is a global optimum as $\frac{\partial U_i}{\partial b_i} > 0$ when $b_i < (1+\alpha) s_i$ and $\frac{\partial U_i}{\partial b_i} < 0$ when $b_i > (1+\alpha) s_i$. The solution for $b_j = (1+\alpha) s_j$ follows from symmetry.

binds. Substituting the equilibrium conditions into Equations 19.3 and 19.4 gives $\frac{\partial U_i}{\partial b_i^1} = s_i \left(\frac{1-\alpha}{1+\alpha}\right) \leq 0 \; \forall \; s_i$ and $\frac{\partial U_i}{\partial b_i^2} = s_i \left(\frac{2\alpha-1}{1+\alpha}\right) \geq 0 \; \forall \; s_i$. By symmetry the same is true for bidder j and the strategies constitute an equilibrium. Implicit bundling occurs as winning one unit when $b_i^1 > b_j^2$ implies that $b_i^2 > b_j^1$ also, so bidders always win both units or neither (as bids are continuous and increasing, the probability of ties is zero). ■

As with the discriminatory auction, the Vickrey auction leads to implicit bundling, as one bidder wins both units in equilibrium. A revenue comparison can be made to the second-price auction for the bundle, in which it is an equilibrium to bid $b_i = (1 + a)\, s_i$. The bidder with the highest signal wins both units and pays the opponent's bid, $(1 + a)\, s_j$. In the Vickrey auction the bidder with the highest signal wins both units and pays the sum of the opponent's bids $b_j^1 + b_j^2 = \frac{1+\alpha}{2} s_j + \frac{1+\alpha}{2} s_j = (1 + a)\, s_j$. Therefore the Vickrey auction generates the same actual revenue as the second-price auction for the bundle if bidders have increasing marginal valuations.

The winner's curse is very strong when bidding for a first unit but weak when bidding on a second unit. Therefore players bid aggressively for their second units, except in this case they are constrained to bidding $b_i^1 = b_i^2$. In the case of constant marginal valuations it was shown that the Vickrey auction generates a continuum of symmetric equilibria. When marginal valuations are increasing, $b_i^1 = b_i^2$ restricts bids to the *fully* symmetric equilibrium, which is unique. There are additional asymmetric equilibria, corresponding to the asymmetric equilibria of the second-price, sealed-bid auction for the bundle.

19.3 The Uniform Price Auction

The uniform price auction has multiple equilibria, just as when bidders have constant marginal valuations. The first corresponds to the second-price auction for the bundle, where in equilibrium $b_i^1 = b_i^2 = \frac{1+\alpha}{2} s_i \; \forall \; i \;,\; s_i$ so a bidder never wins a single-unit. This means that the second bid never affects the price paid for the first unit, so in equilibrium there is no *demand reduction* and the outcome corresponds to that of the Vickrey auction and the second-price auction for the bundle. The proof that this is an equilibrium follows from Proposition 19.2 which demonstrates that there is a symmetric equilibrium of the Vickrey auction in which $b_i^1 = b_i^2 \; \forall \; (s \;,\; i)$ and a straightforward extension of Proposition 16.3 which shows that an equilibrium of the Vickrey auction in which $b_i^1 = b_i^2 \; \forall \; (s \;,\; i)$ is also an equilibrium of the uniform price auction.

A second equilibrium is that of *full demand reduction* which occurs when players bid $\frac{s_j+1}{2}$ for a first unit and zero for a second unit.

Proposition 19.3 *When bidders have increasing marginal valuations it is a Nash equilibrium of the highest losing bid uniform price auction for bidders to bid* $b_i^1 = \frac{s_i+1}{2}$ *and* $b_j^1 = \frac{s_j+1}{2}$ *for first units and* $b_i^2 = b_j^2 = 0$ *for second units providing* $\alpha < 2$.

Proof. Assume that $b_j^1(s_j) = \frac{s_j+1}{2}$ and $b_j^2(s_j) = 0$. As $\frac{s_i+s_j}{2} - b_j^2 = \frac{s_i+s_j}{2} - 0 > 0$ $\forall\ s_i\ ,\ s_j$, any bid b_i^1 for i which always wins the first unit is optimal. If bidder i bids 0 for the second unit, both bidders receive a payoff of $\frac{s_i+s_j}{2}$. Any bid b_i^2 that wins an additional unit always reduces bidder i's payoff as $(1+\alpha)\frac{s_i+s_j}{2} - 2b_j^1 = (1+\alpha)\frac{s_i+s_j}{2} - s_j - 1 < \frac{s_i+s_j}{2}$ providing $\alpha < 2\left(\frac{1+s_j}{s_i+s_j}\right)$ $\forall\ s_i$ Any positive bid b_i^2 that does not win an additional unit is weakly dominated by $b_i^2 = 0$ as increasing b_i^2 above 0 only increases the amount paid on the first unit, directly reducing surplus. Therefore $b_i^2 = 0$ weakly dominates any $b_i^2 = 0$ when the opponent bids $b_j^1(s_j) = \frac{s_j+1}{2}$ for a first unit. ■

This Nash equilibrium of the uniform price auction generates no revenue for the auctioneer, as bidders engage in *full demand reduction*, submitting zero bids for the second unit and therefore the highest losing bid is zero. The revenue equivalence theorem does not hold in this case as the interim allocation and payments differ from the implicit bundling which arises in the other auctions considered in this section. As this equilibrium maximises expected surplus to bidders, it could be argued that it is focal, making the uniform price auction undesirable. This exists providing $\alpha < 2\left(\frac{1+s_j}{s_i+s_j}\right)$ $\forall\ s_i$ and hence that $1 < \alpha < 2$ (by putting the maximum $s_i = 1$ into the constraint). If $\alpha > 2$ then this extreme case of *demand reduction* cannot be an equilibrium, as bidders with high signals would compete aggressively for the second unit.

19.4 Discussion

When bidders have increasing marginal valuations it has been shown that in the Vickrey and discriminatory auctions the constraint $b_i^1 \geq b_i^2$ binds for all bidders and signals. This is sufficient to generate implicit bundling, so many of the results in this chapter are similar to those in the constant marginal valuations case. It can be conjectured that in this simplified setting these results continue to hold even if bidders were free to submit combinatorial bids for the bundle, as providing $b_i^1 \leq b_i^2$ for both signals and bidders, implicit bundling still occurs and the equilibrium requirement is only that $b_i^1 + b_i^2$ satisfies the equilibrium conditions for a single-unit auction of the bundle.

Unlike in the case of constant marginal valuations, the Vickrey auction now has a unique symmetric equilibrium when bids are continuous, increasing functions of signals[2]. All equilibria of the Vickrey and discriminatory auctions involve implicit bundling which is efficient when marginal valuations are increasing. The symmetric equilibrium of the uniform, Vickrey and discriminatory auction are equivalent to the second- and first- price auctions for the bundle respectively, and are therefore revenue equivalent, generating $E[R] = E\left[2\max\left(\left(\frac{1+\alpha}{4}\right)s_i\right)\right] = \frac{1+\alpha}{3}$. This provides an upper bound on the revenue generated in the asymmetric equilibria of the Vickrey auction.

The *full demand reduction* equilibrium of the uniform price auction, that was developed when bidders have constant marginal valuations, exists even when marginal valuations are increasing providing $\alpha < 2$. This is the only equilibrium of any auction discussed in this chapter which leads to an inefficient allocation, in which each bidder receives one unit. In addition, this generates zero revenue. When $\alpha > 2$ this equilibrium is broken, as the gain from winning a second unit outweighs the effects of *demand reduction* and the auction has the same equilibrium as the Vickrey auction.

Claim 19.1 ***Efficiency:*** *When $1 < \alpha < 2$ bidders have increasing marginal valuations and the bundled, discriminatory and Vickrey auctions are efficient. The uniform price auction has both efficient and inefficient equilibria.*

Claim 19.2 ***Expected Revenue:*** *When $1 < \alpha < 2$ bidders have increasing marginal valuations and the auctions can be ranked by expected revenue as follows:*

$$\begin{aligned} ER\,[\textit{first-price bundle}] &= ER\,[\textit{Discriminatory}] \geq \\ ER\,[\textit{second-price bundle}] &= ER\,[\textit{Vickrey}] \geq \\ &\quad ER\,[\textit{Uniform}] \end{aligned}$$

The first inequality binds when the equilibrium of the Vickrey or second-price auction is symmetric. The second inequality binds in the equilibrium in which there is no demand reduction.

[2]There remains a continuum of asymmetric equilibria corresponding to the asymmetric equilibria in the second-price sealed-bid auction for the bundle.

Chapter 20

Relation to the Multi-Unit Auction Literature

This chapter will relate the model to the literature on multi-unit discriminatory, uniform price and Vickrey auctions, much of which focuses on the case of independent private valuations. An extension to the Ausubel auction will also be discussed.

Uniform Price Auctions

The existence of *demand reduction* in uniform price auctions is well documented in continuous *share auctions* in which bidders compete for a share of a unit. Wilson (1979) shows that when bidders do not have private information, there is an infinite number of linear equilibria involving implicit collusion by the bidders. For example, two bidders could divide the object between them by each submitting high bids for the first half of the unit and bidding the reserve price on the marginal share. Bidding above the reserve price means raising the clearing price without increasing a bidder's own share. This is related to the concept of market power: by submitting a very steep demand curve, a bidder makes the opponent a residual monopsonist over the remaining share. The opponent then has an incentive to submit a downward sloping demand curve in a Nash equilibrium. Back and Zender (1993) develop the framework when bidders have common values, by considering equilibria robust to supply uncertainty introduced by non-competitive bidders. These equilibria are independent of private signals and are non-linear. The presence of uncertainty reduces the scope for *demand reduction*. Back and Zender also consider the case where the seller can decrease supply after observing bids, potentially leading to high prices if bidders submit very steep demand curves and so curtailing collusive behaviour. This reduces, but does not eliminate the potential for implicit collusion. Li Calzi and Pavan (2002) show that declaring a supply schedule that is increasing in the price can reduce the steepness of the residual supply curve for any one bidder, encouraging more aggressive bidding and enhancing revenue.

Turning to the discrete case, Ausubel and Cramton (2002) show that when bidders have independent private values, in highest losing bid uniform price auctions it is a weakly dominant strategy to bid their valuation on the first unit. However, there is an incentive to reduce bids on additional units because there is a positive probability that these will set the price of all earlier units. This underlies the analysis of *demand reduction* developed in this thesis, which extends the result of Engelbrecht-Wiggans and Kahn (1998), where bidders may submit zero bids on their second units, to a situation where bidders have interdependent values. Ausubel and Cramton (1998) show that if bidders have constant marginal valuations and affiliated signals then there is an equilibrium in which they submit flat demand curves, consistent with the findings in Chapter Seventeen. In this case the linkage principle (Milgrom and Weber, 1982) holds and second price auctions generate more revenue than first-price auctions. A contribution of this thesis was to show that when bidders have diminishing marginal valuations, the opposite is true: low price equilibria are unique among the class of continuous, weakly increasing equilibria because the auction decomposes into asymmetric single-unit auctions, creating an implicit almost common values problem. Therefore the winner's curse is greatly amplified, leading to low expected revenue. Even slight diminishing marginal valuations leads to an extreme version of Ausubel's (2004) *champion's plague*; when bidders have common values and affiliated signals a bidder's expected value conditional on winning is decreasing in the number of units won.[1]

Generally these extreme theoretical results stand in contrast to the experimental evidence and empirical analysis of treasury auctions. Chapter Twenty-One will argue that bounded rationality may mitigate the theoretical results in some cases, but will reinforce them in others.

Discriminatory Auctions

Under constant marginal valuations the discriminatory auction model considered in Chapter Fifteen generates a unique equilibrium where bidders submit equal bids on all units; this result persists in the case of N bidders, M units and independent private valuations.[2] As bid curves are flat each bidder either wins all or none of the units and it is necessary to condition on winning all the units to avoid the winner's curse.

[1] Ausubel (2004) assumes constant marginal valuations (up to a capacity constraint) so almost common values problems do not arise in his model.

[2] See Lebrun and Tremblay (2003), when in the M unit case bidders have binary valuations.

Engelbrecht-Wiggans and Kahn (1998) argue that while bidders submit demand curves which are steeper than their valuations in the uniform price auction, in the discriminatory auction they minimise their expected payments by flattening their demands, aiming to submit bids just above the expected clearing price. Engelbrecht-Wiggans and Kahn show that when bidders have diminishing marginal valuations, bidders must submit the same bids for both units over some range of signals and different bids for other ranges. Intuitively this is because when bidders have independent private valuations the weaker bidder bids more aggressively. In contrast, in the common value model presented in this thesis the weaker bidder bids more conservatively because of the winner's curse, so when bidders have diminishing marginal values demand curves slope downwards as $b_i^1 \geq b_i^2 \ \forall \ (s \ , \ i)$.

Vickrey Auctions

When bidders have private valuations, the Vickrey auction gives an incentive for bidders to reveal their true valuations and may increase bidder participation. It remains robust against *demand reduction*, as bidders never pay their own bids and so do not stand to benefit by reducing bids on later units, and is solvable by weakly dominant equilibrium strategies. Generally, the problems of multi-dimensional signals (see Jehiel and Moldovanu, 2001) do not extend to the formal model of this thesis.[3] However, in the common values case with decreasing marginal valuations, it was shown in Chapter Eighteen that the Vickrey auction leads to low revenue, because of the almost common values problem which arises implicitly. As well as a lack of robustness to small payoff asymmetries in common value auctions, the Vickrey auction may be used less because bidders are reluctant to reveal private information they do not want disclosed in the final market, sacrificing information rent in secondary markets, or because they do not trust the auctioneer not to take advantage of this information.

Ausubel Auctions

The analysis developed in this thesis can be extended to other multi-unit auctions. Ausubel (2004) argues that when bidders have common values and affiliated signals, the Linkage principle (Milgrom and Weber, 1982) means that ascending auctions generate the most revenue even in a multi-unit setting. In the auction Ausubel proposes, the price increases over time and people drop out so quantity demanded falls.[4] When

[3] If implicit bundling occurs a single-crossing property holds. If bidders have diminishing marginal valuations, the almost common values problem leads to an efficient outcome.

[4] When bidders have independent private valuations then bidding their valuation is a weakly dominant strategy in the Ausubel auction when bidders dropping out is not revealed, and survives iterated deletion of weakly dominated strategies if it is revealed.

a bidder is guaranteed to win a unit it is *clinched* and the winner pays the price at which it was clinched.[5] If an Ausubel auction was used in the two unit, two bidder case with diminishing marginal valuations, bidders would reduce their demand from two units to one unit at a low level to avoid the winner's curse. This is consistent with the weaker bidder dropping out early in the almost common values auction (Klemperer, 1997). This means that a unit is *clinched* for the opponent at a low price, and as bidders pay the price at which units are clinched, the Ausubel auction generates low revenue.

[5] For example suppose 3 units are being auctioned to 3 bidders, with demands at the given prices shown below:

Demand	A	B	C
$P = 1$	3	2	2
$P = 2$	2	1	1

The auctioneer increases the price until it reaches $P = 2$, at which point A wins one unit with certainty, "clinching" it, and pays a price of 2. The price then continues to rise to clear the remaining units.

Chapter 21

Empirical Literature, Generalisations and Bounded Rationality

This thesis predicts that when bidders have common values and constant or increasing marginal valuations, they will submit flat demand curves in the Vickrey and discriminatory auctions, and may engage in demand reduction in the uniform price auction, leading to low revenue. This section will argue that flat demand curves are not supported empirically, which could be explained if bidders have decreasing marginal valuations. Even if the payoffs are constant, these could be motivated implicitly as the reduced form of reciprocity considerations or risk aversion. Under decreasing marginal valuations, an implicit almost common values problem means that second price auctions generate very low revenue and *full demand reduction* occurs in the uniform price auction, as bidders submit bids of zero for a second unit. This chapter will review some experimental and empirical results. Some of these might be explained by introducing more bidders or units, while others can only be explained by bounded rationality. It will be argued that bounded rationality could reinforce some of the theoretical results as the equilibrium strategy can be motivated by players who form simple analogies about how the game is played. Both the generalisations and the analysis of bounded rationality lend support to the Milgrom (2004) and Klemperer (2004) doctrine that the *details* of an auction design are extremely important.

21.1 Empirical Results

Much of the literature relating to treasury auctions is aimed at determining whether uniform price or discriminatory auctions raise more revenue. Following Friedman

(1960) claims have been made that the effects of asymmetric information are reduced in the uniform price auction, leading to smaller, uninformed bidders entering and increasing competition.[1] Several studies have been carried out comparing the expected revenue raised by discriminatory and uniform price treasury auctions. Fevrier *et al* (2002) consider French treasury auctions in 1995 and find that discriminatory auctions raise more revenue than uniform price auctions. In addition Hortacsu (2002) examines Turkish treasury auctions and finds that discriminatory auctions raise significantly more revenue. However, there is also evidence reporting ambiguous findings. Malvey and Archibald (1998) find that the switch from discriminatory to uniform price auction for the U.S. Treasury had an insignificant impact on revenue. Nyborg and Sundaresan (1998) find some under-pricing in discriminatory auctions but the difference in average mark ups between the discriminatory and uniform price auctions is insignificant. In treasury auctions, neither extreme low price equilibria of the uniform price auction nor flat demand curves are observed empirically.

There is a fairly large experimental literature on multi-unit auctions when bidders have independent private valuations. The most relevant to this thesis is Engelmann and Grimm (2006) who compare the different auction formats discussed in this thesis in the two unit, two bidder, constant marginal valuations case. In all the auctions, bidders submit downward sloping bids, in contrast to the equilibrium prediction in discriminatory auctions. This is consistent with *demand reduction* in the uniform price auction, although the equilibrium in which bid for a second unit are zero rarely occurs. The discriminatory auction raises the most revenue, although consistent with other experiments (Kagel and Levin, 2001, for example) they frequently find that bids for a first unit are greater than valuations in Vickrey and uniform price sealed-bid auctions, which is a weakly dominated strategy. Although models of bounded rationality have been proposed to explain this overbidding in multi-unit auctions, it seems likely that it is related to experimental studies of Vickrey auctions for a single unit, where bidders often bid above their valuations.[2] For this reason, the Ausubel auction leads to greater efficiency than other mechanisms. In a field experiment, List and Lucking-Reiley (2000) find significantly higher first unit bids in uniform price auctions than in Vickrey auctions (although overbidding occurs in both).

Sade *et al* (2006) run an experiment with multi-unit demand *and* common values. However, there is no private information in their auction, as all bidders know the ex post value with certainty. twenty-four units are allocated to five bidders who submit

[1] Sherman (2002) points out that this could lead to free riding in uniform price auctions, so discriminatory auctions might provide better incentives for information gathering.

[2] See Kagel and Levin (2006).

demands at four specific prices. In contrast to other experiments, they find that the discriminatory auction raises *less* revenue than the uniform price auction. Allowing the seller in uniform price auctions to reduce supply ex post does not have a statistically significant effect on revenue. Goswami *et al* (1996) find that bidders coordinate on underpricing equilibria in uniform price auctions when pre-play communication is possible. This appears to support the idea presented in this chapter that the existence of a simple, low price equilibrium is important for uniform price auctions to generate low revenue. In this paper, however, coordination is facilitated by a lack of private information.

Turning to the experimental evidence for common value auctions for a single unit, the winner's curse is well established when bidders are inexperienced (Kagel and Levin, 2006, provide a recent survey). The effect of *almost common values* are tested experimentally by Avery and Kagel (1997) and Rose and Levin (2005); both find that bids respond only proportionally to the advantage. Avery and Kagel (1997) use a second-price sealed-bid auction and find overbidding even when bidders have common values, because they appear not to account for the winner's curse. Both papers find that bids are closer to the expected value than they are to the equilibrium.[3]

21.2 Generalisations

This section will investigate whether different information structures or additional bidders and units could explain the empirical results. Section 21.3 will argue that as well as having additional explanatory power, bounded rationality will become even more important when large numbers of bidders and units increases the complexity of the situation.

Different Information Structures

Although a modified *wallet game* was used to simplify the analytical model and calculate expected revenue, many of the results in this thesis would generalise to other information structures when there are two bidders and two units. Equilibria involving flat demands when bidders have constant marginal valuations also occur when they have independent private values or common values with affiliated signals (see Chapter Twenty). When bidders have decreasing marginal valuations, the almost common values problem occurs generally.

[3] Rose and Levin (2005) attempt to model heterogeneity between bidders, but find firstly that few bidders are *sophisticated* by their classification (i.e. they always win when they are advantaged and always lose when they are disadvantaged). In addition, even these bidders are not very *sophisticated*.

Additional Bidders

When there are two units but more than two bidders the extreme almost common value theoretical prediction of very low revenue need not hold. Levin and Kagel (2005) model almost common value single-unit auctions and show that introducing more than one disadvantaged bidder mitigates the result. The intuition is that when one disadvantaged bidder wins the auction, he may have tied against the other disadvantaged bidder, reducing the extreme effect of the winner's curse. To some extent this may be offset because the winner's curse becomes more powerful as more bidders are introduced (Bulow and Klemperer, 2002).

Additional Units

The model extends to the case where there are two bidders and more than two units. Implicit bundling will still occur in the discriminatory auction and the fully symmetric equilibrium of the Vickrey auction when marginal valuations are constant or increasing. However, the zero bid equilibrium of the uniform price auction will be broken when bidders compete symmetrically for the marginal unit (for example when there are two bidders and three units). When bidders have diminishing marginal valuations, a special case provides some intuition. The solution for the discriminatory auction in Proposition 18.1 can be applied when the value of additional units to each bidder takes the form of a harmonic series, so the first is worth $\frac{s_1+s_2}{10}$, the second $\frac{s_1+s_2}{11}$ a third $\frac{s_1+s_2}{12}$ and so on.[4] In this case the discriminatory auction continues to generate a higher expected revenue than the bundled auction. The Vickrey auction will lead to low revenues as bidders compete aggressively for the first half of the units and submit low bids for the second half of the units. There is non-monotonicity in revenue as the number of units increases in the Vickrey auction, because more revenue is raised when there is an odd number of units as bidders compete on the margin. This effect leads to larger non-monotonicities in the uniform price auction, because if bidders compete for the marginal unit, this affects the price of all units. Submitting zero bids is no longer an equilibrium when the number of units is odd, although bids for the marginal unit would still be low because of *demand reduction.*[5]

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[4]This is because the maximum bid on the first and third units are $\frac{\alpha}{1+\alpha} = \frac{1}{10}\frac{\frac{10}{12}}{1+\frac{10}{12}} = \frac{1}{22}$ and the maximum bid for the second unit are also $\frac{\alpha}{1+\alpha} = \frac{1}{11}\frac{1}{1+1} = \frac{1}{22}$. Marginal bids are weakly constant only if $s_i = 1$ or $s_i = 0$ and the auction decomposes. If this special case does not hold, then bidders will submit flat bid curves with positive probability, in which case the bundled auction could well generate greater revenue.

[5]In the harmonic structure listed above bundle will generate strictly higher expected revenue revenue as $E\left[R_{bundle}\right] = \frac{1}{3}\left[\frac{1}{10} + \frac{1}{11} + \frac{1}{12}\right] > 3\left[\frac{1}{3}\frac{1}{11}\right] = 3E\left[R_{marginalunit}\right]$

Summary

The result that bidders submit flat demands when they have constant marginal valuations is robust to changes in both the number of bidders and the number of units and different information structures. Submitting a downward sloping demand curve only hedges against the winner's curse *if other bidders also submit downward sloping demand curves*, which is not the case in the symmetric equilibria of the discriminatory and Vickrey auctions. One method of overcoming this is to introduce bounded rationality directly, while another is to assume decreasing marginal valuations. In the latter case, the theory suggests that considerable non-monotonicities may arise in uniform price auctions as the number of bidders and units changes. If bidders are symmetric, then when the number of units is perfectly divisible by the number of bidders, the uniform price auction is expected to generate low revenue. Introducing elements of bounded rationality would reinforce this result, as coordination in the uniform price auction would involve simple bidding strategies.

21.3 Bounded Rationality

The rationality of bidders underlies the theoretical analysis of auction theory. Elements of bounded rationality can be introduced in two areas: firstly, in common values auctions, bidders may fail to properly account for the winners curse when forming their bids. Secondly, even if bidders correctly condition their valuations on winning the objects, calculating their bids may be complex (especially as the opponent's bidding strategy is uncertain). This is important in multi-unit auctions where the allocation and payment rules are complicated. Even a bidder who fully understands the mechanism would bid differently if he thought that his opponent did not, so these issues are clearly important when designing a multi-unit auction.

Complexity

Two papers which focus on different aspects of the complexity involved in bidding in multi-unit auctions are Goldreich (2004) and Kagel and Levin (2006). Goldreich (2004) provides some evidence of bounded rationality in Treasury auctions. Treasury auctions are framed in yield space, and the investors appear to use a yield bidding heuristic to calculate their optimal bids. This is suboptimal because yields are converted to price space, which is coarser. Frequently a situation arises where a bidder could have increased his yield (and improve his chance of winning) without increasing the price he would pay if his bid was marginal. Goldreich finds that 28% of bids in discriminatory and 52% of bids in uniform price auctions are dominated. As using dominated bids is less costly in uniform price auctions this could be seen as evidence that people behave more suboptimally when it is less costly to do so, and related to the notion of optimality based refinement in Chapter Ten.

In an experimental study, Kagel and Levin compare the Vickrey and Ausubel auctions when bidders have independent private valuations and multi-unit demands.[6] They find that bidding behaviour is significantly closer to the true equilibrium (bidding the true valuation) in the Ausubel auction when bidders are informed when their opponents drop out and when clinching occurs. They conclude that although the solution concept in this case is only *iterated deletion of weakly dominant strategies*, rather than *dominant strategies*, when drop-out information is provided the Ausubel auction is more transparent and boundedly rational bidders are able to submit bids that are closer to the equilibrium (their true valuations).

Complexity is closely linked to the analysis of the low price equilibria of the uniform price auction analysed in the previous section. When bidders are symmetric and the number of units is perfectly divisible by the number of bidders, it seems reasonable that they coordinate on a very simple equilibrium, in which each bidder receives an equal share. Levin (2004) proposes that bids above valuations for a first unit could reinforce an equilibrium of low bids for a second unit. Section 21.1 discussed Sade *et al* (2006) in which 24 units were auctioned between 5 bidders. Introducing supply certainty had a statistically insignificant effect on revenue. It could be conjectured that this is because there is sufficient strategic uncertainty already; if there were 25 units (or 4 bidders) then an equal split of 5 units (or 6 units) each is not only focal, but also simple to implement. In this respect, introducing bounded rationality reinforces the analysis of the uniform price auction, and the importance of it being impossible to share the units evenly in raising revenue.

The Winner's Curse

Another explanation for submitting bids above valuations in the Vickrey and uniform price auctions is that bidders suffer from the winner's curse.[7] Eyster and Rabin (2005) introduce the concept of *cursed equilibrium*, in which bidders form correct beliefs about the opponent's behaviour but do not fully appreciate that their opponent's actions depend on their private information. In the *wallet game* considered in this paper, a

[6] Technically only a single human bidder has multi unit demand and competes against four computers bidding equilibrium strategies for single units.

[7] Figure 18.3 illustrates the adjustment necessary to avoid the winner's curse in the discriminatory auction. In contrast to the results in second price auctions, in this case failure to properly adjust for the winner's curse would lead to overbidding at all signals.

cursed equilibrium involves players calculating their valuations as $V = \frac{s_1+(1-\lambda)s_2+\frac{\lambda}{2}}{2}$. λ represents the extend of the curse; if $\lambda = 1$ then the auction is *fully cursed* and bidders take the unconditional expected value of their opponent's bids, so $V = \frac{s_1+\frac{\lambda}{2}}{2}$. The standard Bayesian Nash equilibrium arises if $\lambda = 0$. A problem with this approach is explaining how bidders learn sufficiently to have correct expectations about their opponent without also learning to avoid the winner's curse. Jehiel and Koessler (2006) propose an analogy-based alternative in which players bundle states into analogy classes and form average beliefs about their opponent's behaviour in different states. The *fully cursed equilibrium* is equivalent to their *private information analogy partition* in which players assume that given their private information, their opponent behaves in the same way in all possible states. Allowing arbitrary analogy partitions could lead to players partially adapting to the winner's curse if they form less coarse expectations about the opponent's behaviour, rather than taking a convex combination of the full information and fully cursed equilibria. Unlike the approach proposed in Part Two of this thesis however, analogy classes would be given exogenously as part of the strategic environment.

Crawford and Iriberri (2006) propose an alternative concept of *level-k thinking* to explain the winner's curse.[8] *Level*-0 bidders submit random, uniformly distributed bids over the observed range.[9] *Level*-1 bidders best respond to this behaviour, while *Level*-2 bidders best respond to the *level*-1s. As *level*-0s bid randomly, *level*-1s will correspond to Eyster and Rabin's *fully cursed* bidders (although the *level-k* approach differs as it is not based on an equilibrium). Using the second price auction experimental data of Avery and Kagel (1997), Crawford and Iriberri find that the majority of bidders are *level*-1. Applied to the information and payoff structure of the *wallet game*, *level*-1s will overbid conditional on a low signals and underbid when they receive a high signal.

If bidders do not take full account of the winner's curse, the extreme effect of almost common values will be mitigated, as disadvantaged bidders may submit moderate bids even in second price auctions, increasing expected revenue. While this may be the case for the Vickrey auction (see List and Lucking-Reiley, 2000), in the uniform price auction, decreasing marginal valuations may still play an important role, even if the almost common values effect is proportional.[10] This is because when the number of units is perfectly divisible by the number of bidders, decreasing marginal valuations may aid coordination by making an equal split even more focal.

[8] Stahl and Wilson (1994) introduce this approach in games of complete information.

[9] They also allow for a *truthful level*-0 type who bids his own valuation.

[10] As in Avery and Kagel (1997) and Rose and Levin (2005), discussed in Section 21.1.

Non-Strategic Bidders

Even if most bidders are able to fully comprehend the multi-unit auction, overcoming both complexity and the winner's curse, a few smaller bidders who do not may significantly affect the ability to coordinate on an implicitly collusive equilibrium. The non-strategic bidders of Back and Zender (1993) could be motivated as being boundedly rational, as submitting very high bids for n units in a uniform price auction is strategically equivalent to a bidder choosing a priori to buy n units at the clearing price (which the auctioneer guarantees to sell). Therefore a few boundedly rational bidders could introduce the supply uncertainty necessary to avoid extreme implicit collusion and very low revenues.

Chapter 22

Conclusion

The theory of multi-unit demand, common value auctions is relatively underdeveloped. In this thesis a modified *wallet game* was used to analyse the two bidders, two units case. Both bidders derive the same utility $V_i^1 = \frac{1}{2}(s_i + s_j)$ from winning a single-unit and $V_i^1 + V_i^2 = \frac{1+\alpha}{2}(s_i + s_j)$ from winning two units. Chapter Sixteen showed that the discriminatory and Vickrey auctions can decompose into two single-unit auctions, as for either bidder to win a second unit, he competes against his opponent's bid for a first unit. First order conditions were derived for the highest losing bid uniform price auction and the first-price, sealed-bid auction for the bundle of units.

The next chapters analysed the equilibria of these auctions for different values of α. In Chapter Seventeen, $\alpha = 1$ so bidders had constant marginal valuations as $V_i^1 = V_i^2 = \frac{s_i+s_j}{2}$. When $\alpha < 1$, in Chapter Eighteen, bidders had diminishing marginal valuations as $V_i^2 < V_i^1$. Finally in Chapter Nineteen, when $\alpha > 1$, bidders had increasing marginal valuations as $V_i^2 > V_i^1$. Revenue rankings and efficiency implications were analysed for each situation and auction mechanism. Under constant or increasing marginal valuations, equilibria of the Vickrey and discriminatory auctions involved bidders submitting flat demand curves ($b_i^1 = b_i^2$). This led to implicit bundling, as the bidder with the highest signal would win all the units (an efficient allocation), and revenue equivalence with the auction for the bundle. The uniform price auction had multiple equilibria, one of which corresponded to the implicit bundling equilibrium of the Vickrey auction, and another which involved *full demand reduction*, where bidders submitted high bids for their first unit and zero bids for a second.

As flat (or increasing) demand curves are not supported empirically, Chapter Eighteen analysed the case of decreasing marginal valuations. The uniform price and Vickrey auctions allocated the units efficiently, one to each bidder, but decomposed into asymmetric single-unit auctions, creating an implicit almost common values problem in which the winner's curse is severe, leading to low revenue. The discriminatory auction could lead to inefficient allocations, but generated greater expected revenue than the uniform price, Vickrey and bundled auctions. The latter result was because the impact of the winner's curse meant that the possibility of capturing the additional gross surplus of an efficient allocation outweighed the increased competitiveness of the symmetric auction for the bundle.

Chapter Twenty-One considered some empirical results from experimental economics and treasury auctions. The result that bidders with constant marginal valuations submit flat demands was robust to changes in the information structure and number of bidders and units, which motivated decreasing marginal valuations.[1] While some empirical results could be explained by introducing more units and bidders, others, such as overbidding on first units, could only be explained by introducing bounded rationality. Bidders could fail to properly account for the winner's curse or calculate optimal bids due to the uncertainty and complexity of the environment. Even if most bidders are able to fully comprehend the multi-unit auction, a few boundedly rational smaller bidders could introduce supply uncertainty which would significantly affect the ability of larger bidders to collude implicitly in the uniform price auction.[2]

Generalisation showed that expected revenue in the uniform price auction was non-monotonic as the number of bidders and units changed, as expected revenue was low when the number of units was perfectly divisible by the number of bidders. Introducing bounded rationality reinforced this result, as coordination involved simple bidding strategies, submitting high bids for an equal share of the units and low bids for the remainder. In situations where the number of units was not divisible by the number of bidders, both the formal theory and bounded rationality (because of greater complexity) suggested that revenue in the uniform price auction might be closer to that in the discriminatory auction, which had no low bid equilibrium nor an implicit almost common value problem. Both the generalisations and the analysis of bounded rationality lend support to the Milgrom (2004) and Klemperer (2004) argument that the *details* of an auction design are extremely important.

[1] Even if payoffs are constant, decreasing marginal valuations could be motivated implicitly as the reduced form of reciprocity considerations or risk aversion.

[2] This provides an alternative motivation for the non-strategic bidders in Back and Zender (1993).

Part IV

General Conclusion, Bibliography and Appendices

Chapter 23

General Conclusion

This thesis has argued that introducing bounded rationality can provide additional insight in a range of microeconomic situations. In every part, bounded rationality is related to the way forming *categories* or *analogies* affects players' beliefs. In the first part, a screening process was analysed using a model in which information acquisition constraints cause projects to be evaluated using coarse *categories*. This explained some empirical results on résumé screening, which are particularly relevant as both coarse signals and screening are best motivated early in the search process. In the second part, players formed endogenous *analogy*-based expectations of their opponent's behaviour. This could be used to explain why players *pass* for long periods and mix latterly in games of complete and perfect information such as the Centipede game. Finally, the third part found that decreasing marginal valuations can generate theoretical results consistent with the empirical observation that bidders submit downward sloping demand curves in multi-unit auctions. Additional insights were provided by considering the case when players formed *analogies* that their opponent's bidding strategy was independent of his private information. The theoretical results on low price equilibria in uniform price auctions were reinforced if players were able to coordinate on an equilibrium using the analogy with the case of an equal division of the units.

This first part of the thesis introduced bounded rationality into a decision problem, by investigating how a decision maker might optimally screen projects to choose one that maximises expected utility. A screening heuristic was motivated early in the search process, where information acquisition constraints lead to coarse information and it is not yet optimal to use a more accurate but costly procedure involving sequential search or pairwise comparisons. It was shown that introducing asymmetry (or bias) in the screening process reduced the *a priori* expected cost of errors in the decision stage. Fully asymmetric screening was related to optimal sequential search, but required that projects be uniquely identifiable and a different threshold must be memorised for every one. In partially asymmetric screening, the decision maker used an exogenous characteristic to identify projects, and assessed projects in different groups using different thresholds. Optimal partially asymmetric screening used a lower screening threshold for the minority group. However, projects in the majority group had a higher probability of being chosen *a priori*, because conditional on passing they were chosen in preference to projects in the minority group in the decision stage.[1] This insight was related to *biased screening* models in the literature on the economics of discrimination. As well as demonstrating the optimality of asymmetry in a class of communication problems, this thesis shows *minority discrimination* could be optimal in a screening process without *any* initial exogenous differences between groups.

The second part of the thesis investigated how bounded rationality might lead to players forming analogies about how their opponents would move, which could overcome the *finite horizon paradoxes* that arise in a generalised Centipede game. In an analogy-based expectations equilibrium (Jehiel, 2005), players bundle nodes at which their opponents move into analogy classes, and form expectations that the opponents behave in the same way within each class. This thesis investigated which analogy-based expectations equilibria are robust if players formed analogy classes *endogenously*, so they are more likely to form analogies when the opponent's behaviour is similar, and form analogies more carefully when suboptimal actions would prove costly. The first refinement is similar to the idea of consistency underlying the approach, while the second allowed the robustness of an analogy class to be linked to the payoffs specifying the game; both refinements reduced the set of robust analogy classes and led to similar restrictions on behavioural strategies. It was demonstrated that if passing were to be sustained in equilibrium, then analogy classes in which players expect their opponent to mix were dramatically more robust than those in which they use pure strategies, as one player would always observe his opponent *take* at a node in which he expected the opponent to *pass* with high probability. This intuition was extended to the case when players form multiple analogy classes, where this thesis proposed an intuitive solution to the finite horizon problem: an equilibrium consists of players passing for a given number of nodes and then both mixing towards the end of the game.

The final part of the thesis developed a theoretical model of common value auctions in which bidders have multi-unit demand. Bounded rationality helped to explain empirical and experimental results (multi-unit auctions are often complex) as well as reinforcing some of the theoretical results. When bidders had constant or increasing

[1] If the grouping was endogenous, it was optimal to divide the projects into a majority and minority rather than two groups of equal sizes.

marginal valuations the discriminatory, Vickrey and uniform price auctions had an equilibrium in which bidders submitted flat demand curves.[2] This led to implicit bundling, as the bidder with the highest signal would win all the units (an efficient allocation), and revenue equivalence with the auction for the bundle. This is consistent with several other papers, but is not supported empirically, and this motivating the use of decreasing marginal valuations.[3] In this case the uniform price and Vickrey auctions allocated the units efficiently, but decomposed into asymmetric single-unit auctions, creating an implicit *almost common values problem* leading to low revenue. The discriminatory auction could allocate the units inefficiently, but led to greater expected revenue than the uniform price, Vickrey and bundled auctions. While some empirical results could be explained by introducing more units and bidders, others, such as overbidding on first units, could only be explained by introducing bounded rationality. Bidders could fail to properly account for the winner's curse or calculate optimal bids due to the uncertainty and complexity of the environment. Even a few boundedly rational, smaller bidders could introduce supply uncertainty which would significantly affect the ability of the larger bidders to collude implicitly in the uniform price auction.[4] Generalisation of the model showed that expected revenue in the uniform price auction was non-monotonic as the number of bidders and units changed, as expected revenue was low when the number of units was perfectly divisible by the number of bidders. Introducing bounded rationality reinforced this result, as coordination involved simple bidding strategies, submitting high bids for an equal share of the units and low bids for the remainder. In situations where the number of units was not divisible by the number of bidders, both the formal theory and bounded rationality (because of greater complexity) suggested that revenue in the uniform price auction might be closer to that in the discriminatory auction, which had no low bid equilibrium nor an implicit almost common value problem. Both the generalisations and the analysis of bounded rationality lend support to the Milgrom (2004) and Klemperer (2004) argument that the *details* of an auction design are extremely important.

[2] The uniform price auction also has an equilibrium of full demand reduction in which bidders submit zero bids for their second units.

[3] Even if payoffs are constant, decreasing marginal valuations could be motivated implicitly as the reduced form of reciprocity considerations or risk aversion.

[4] This provides an alternative motivation for the non-strategic bidders in Back and Zender (1993).

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Appendix A

Asymmetric Screening

A.1 Optimal Partitions are Intervals

Appendix A.1 sketches the proof that the decision maker should always use intervals rather than partitions. Consider the following partitions on a random variable X_i.

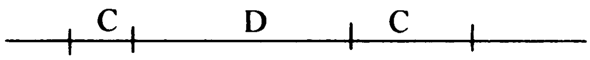

Figure A.1

Here the signal communicated is that X is moderate, D, or extreme, C. Assume that $E[C] \geq E[D]$ (the proof extends to the opposite case) and that the problem is not degenerate, so there are some signal profiles in which C is chosen but D would not be. This is equivalent to stating that $\Pr[X_i \text{ chosen } \mid C] > \Pr[X_i \text{ chosen } \mid D]$.

Following Dow (1991) the partition C can be made into an interval by moving the probability mass from the lower section of the partition to the bottom of the upper end of the partition. This is illustrated in Figure A.2.

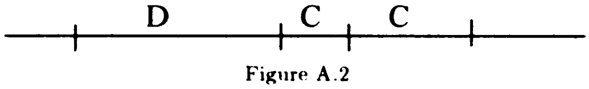

Figure A.2

Assume that conditional on the signal profile, the decision rule does not change (reoptimising may lead to further gains). The new partitions of X_i can be decomposed as in Figure A.3:

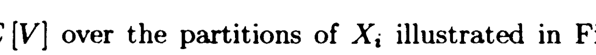

Figure A.3

Decomposing $E[V]$ over the partitions of X_i illustrated in Figure A.3 gives the same $E[V \mid D^*]$ and $E[V \mid C^*]$ as in Figure A.1. Likewise, expected value conditional on X_i falling in the unchanged partitions above C^* and below D' is the same. The change in expected value between the partitions illustrated in Figures A.1 and A.3 will depend only on $E[V \mid D']$ and $E[V \mid C']$.

$$
\begin{aligned}
\Delta E[V] &= E[D']\Pr[D']\{\Pr[X_i \text{ chosen } \mid D] - \Pr[X_i \text{ chosen } \mid C]\} \\
&\quad + E[C']\Pr[C']\{\Pr[X_i \text{ chosen } \mid C] - \Pr[X_i \text{ chosen } \mid D]\} \\
&= \Pr[C'][\Pr(X_i \text{ chosen } \mid C) - \Pr(X_i \text{ chosen } \mid D)][E[C'] - E[D']]
\end{aligned}
$$

because $\Pr[D'] = \Pr[C']$ by construction. This means that $\Delta E[V] > 0$ as $\Pr(X_i \text{ chosen } \mid C) > \Pr(X_i \text{ chosen } \mid D)$ and $E[C'] > E[D']$. Therefore this change increases $E[V]$.

A.2 Derivation of Optimal Communication by Minimising Expected Error

Appendix A.2 gives the derivation of expected error in the case when $b \geq a$ as an example of Proposition 3.2 which shows that minimising expected error is equivalent to maximising $E[V]$ directly because $E[V] = E[V^*] - E[\varepsilon]$, where $E[V^*]$ is the full information expected value (a constant) and $E[\varepsilon]$ is the expected error. $E[\varepsilon \mid s\ ,\ \varepsilon > 0]$ is simple to evaluate in the uniform case.

s	Choice	$E[\varepsilon \mid s\ ,\ \varepsilon > 0]$	$\Pr[\varepsilon > 0 \mid s]$	$E[\varepsilon \mid s, \varepsilon > 0]$	$\Pr[\varepsilon \mid s]$	$\Pr[s]$
X_L, Y_L	Y	$E[X - Y \mid Y \geq X, s]$	$\Pr[Y \geq X \mid s]$	$\frac{a}{3}$	$\frac{a}{2b}$	ab
X_L, Y_H	X	$E[Y - X \mid X \geq Y, s]$	$\Pr[X \geq Y \mid s]$	N/A	0	$a(1-b)$
X_H, Y_L	Y	$E[X - Y \mid Y \geq X, s]$	$\Pr[Y \geq X \mid s]$	$\frac{b-a}{3}$	$\frac{b-a}{1-a}\frac{b-a}{b}\frac{1}{2}$	$(1-a)b$
X_H, Y_H	Y	$E[Y - X \mid X \geq Y, s]$	$\Pr[X \geq Y \mid s]$	$\frac{1-b}{3}$	$\frac{1-b}{1-a}\frac{1}{2}$	$(1-a)(1-b)$

Table A2.1: Probability and expected errors under asymmetric communication

Composing $E[\varepsilon]$ from the probabilities and conditional expectations in Table A2.1 gives:

$$
\begin{aligned}
E[\varepsilon] &= \frac{1}{6}\left[a^3 + (b-a)^3 + (1-b)^3\right] \\
&= \frac{2}{3} - \frac{1}{2}\left[1 + ab^2 - a^2 b + b - b^2\right] \\
&= E[V^*] - E[V]
\end{aligned}
$$

As $E[V] = \frac{1}{2}\left[1 + ab^2 - a^2 b + b - b^2\right]$ is derived in Proposition 3.4. Therefore first order conditions give the same conditions on a and b:

$$
\begin{aligned}
\frac{dE[\varepsilon]}{da} &= \frac{1}{6}\left[3a^2 - 3(b-a)^2\right] = \frac{b}{2}[2a - b] \\
\frac{dE[\varepsilon]}{db} &= \frac{1}{6}\left[3(b-a)^2 - 3(1-b)^2\right] = \frac{1}{2}(1-a)[2b - a - 1]
\end{aligned}
$$

Alternatively if the restriction $a = b$ is applied:

$$
\begin{aligned}
E[\varepsilon] &= \frac{1}{6}\left[a^3 + (1-a)^3\right] \\
\frac{dE[\varepsilon]}{da} &= \frac{1}{6}\left[3a^2 - 3(1-a)^2\right] = a - \frac{1}{2}
\end{aligned}
$$

A.3 Optimal Intervals Under Fully Asymmetric Communication

When there are N signals, under optimal fully asymmetric communication each signal X_i is partitioned into intervals by a threshold a_i so that $\{X_{iL}\ ,\ X_{iH}\} = \{[0, a_i)\ ,\ [a_i, 1]\}$. It is assumed that $a_1 \geq a_2 \geq ...a_{N-1} \geq a_N$ without loss of generality (as labelling the projects is arbitrary in this case). $E[V]$ can be derived as follows:

$$\begin{aligned}
E[V] &= E[X_1 \mid X_{1H}] \Pr(X_{1H}) \\
&\quad + \Pr(X_{1L}) E[X_2 \mid X_{2H}] \Pr(X_{2H}) \\
&\quad +...+ \\
&\quad + \Pr(X_{1L}) ... \Pr\left(X_{(N-1)L}\right) E[X_N \mid X_{NH}] \Pr(X_{NH}) \\
&\quad + \Pr(X_{1L}) ... \Pr(X_{NL}) E[X_1 \mid X_{1L}] \\
&= \frac{1-a_1^2}{2} + a_1 \frac{1-a_2^2}{2} + ... + [a_1 a_2 ... a_{n-1}] \frac{1-a_n^2}{2} + [a_1 a_2 ... a_{n-1} a_n] \frac{a_1}{2}
\end{aligned}$$

The first order conditions can then be calculated:

$$\begin{aligned}
\frac{dE[V]}{da_1} &= \frac{1}{2}\left[-2a_1 + 1 - a_2^2 + ... + [a_2 ... a_{n-1}]\left(1 - a_n^2\right) + 2[a_2 ... a_{n-1} a_n] a_1\right] \\
\frac{dE[V]}{da_2} &= \frac{a_1}{2}\left[-2a_2 + ... + [a_3 ... a_{n-1}]\left(1 - a_n^2\right) + [a_3 ... a_{n-1} a_n] a_1\right] \\
&\cdots \\
\frac{dE[V]}{da_i} &= \frac{a_1 a_2 ... a_{i-1}}{2}\left[-2a_i + 1 - a_{i+1}^2 + ... + [a_{i+1} ... a_{n-1}]\left(1 - a_n^2\right) + 2[a_{i+1} ... a_{n-1} a_n] a_1\right] \\
&\cdots \\
\frac{dE[V]}{da_{n-1}} &= \frac{a_1 a_2 ... a_{n-2}}{2}\left[-2a_{n-1} + 1 - a_n^2 + 2a_n a_1\right] \\
\frac{dE[V]}{da_n} &= \frac{a_1 a_2 ... a_{n-1}}{2}\left[-2a_n + a_1\right]
\end{aligned}$$

Therefore the solution which maximises $E[V]$ will be characterised by Equations A.1, A.2 and A.3.

$$a_1 = \frac{1+a_2^2}{2} + \frac{1}{2}[a_1 a_2 ... a_{n-1} a_n] \quad \text{(A.1)}$$

$$a_i = \frac{1+a_{i+1}^2}{2} \quad \text{(A.2)}$$

$$a_n = \frac{a_1}{2} \quad \text{(A.3)}$$

The first order conditions can be substituted back into the expression for $E[V]$ so that when the partitions are formed optimally

$$E[V] = \frac{1}{2}\left(1 - a_1^2 + a_1\left(1 + a_2^2\right)\right)$$

A.4 Switching Intervals About a Common Boundary Does Not Change Expected Utility

Any set of intervals with an interval boundary that is common to the identically independently distributed project values X and Y (ϕ in Figures A.4 and A.5) will give the same expected value if the interval structure after ϕ is switched between signals X and Y. Therefore the two interval structures illustrated in Figures A.4 and A.5 will have the same expected payoffs.

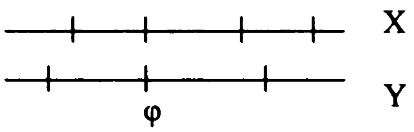

Figure A.4

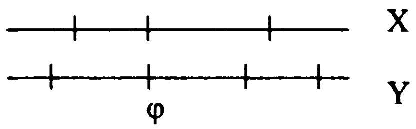

Figure A.5

This follows because the expected utility can be conditioned on X and Y relative to ϕ.

$$\begin{aligned} E[V] = \; & E[V \mid X \leq \phi\,,\ Y \leq \phi]\Pr[X \leq \phi\,,\ Y \leq \phi] \\ & + E[V \mid X \leq \phi\,,\ Y > \phi]\Pr[X \leq \phi\,,\ Y > \phi] \\ & + E[V \mid X > \phi\,,\ Y \leq \phi]\Pr[X > \phi\,,\ Y \leq \phi] \\ & + E[V \mid X > \phi\,,\ Y > \phi]\Pr[X > \phi\,,\ Y > \phi] \end{aligned}$$

The conditional expected values will be the same in all cases. Firstly, if both realisations are below ϕ or both above ϕ, switching realisations between projects will not affect $E[V]$ because X and Y are independently identically distributed (so the intervals are the same in Figures A.4 and A.5 anyway). Secondly if one realisation is below ϕ while the other is above ϕ then the structure of intervals below ϕ is irrelevant, as the higher value will be chosen (the decision is independent of irrelevant alternatives). This argument extends to any number of projects.

A.5 Optimal Internal Intervals Do Not Strictly Contain Intervals on Other Projects

The notation used when there are multiple intervals is that each interval over X and Y is labelled by its expectation. The probability of the realisation of X (or Y) lying in specific partition $A, B, C...$ is given by the lower case letter $a, b, c...$ respectively. Consider the case where an interval on one signal lies within the bounds of an interval on another signal. This is illustrated in Figure A.6 where interval S falls within interval T (also U falls within V).

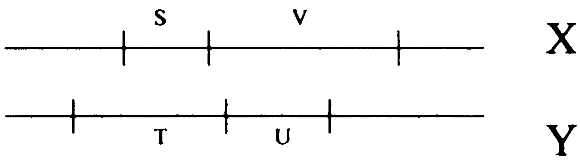

Figure A.6

The intuition in Appendix A.1 suggests this is suboptimal, as conditional on a realisation of signal profile $[S, T]$ the expected values will be closer together than they would if the intervals overlapped. Formally the proof that this is suboptimal involves switching the interval boundaries above and including the upper threshold on S from one signal to the other. This process is illustrated in Figure A.7. The intervals are used to construct the proof are finer than those used by the decision maker to partition signal space which are indicated by the thick black lines. So the decision maker could receive a signal that the realisation of X lies in the set of $[Q \cup F]$ or that the realisation of Y lies in the set of $[C \cup R]$, but not that X lies in Q directly.

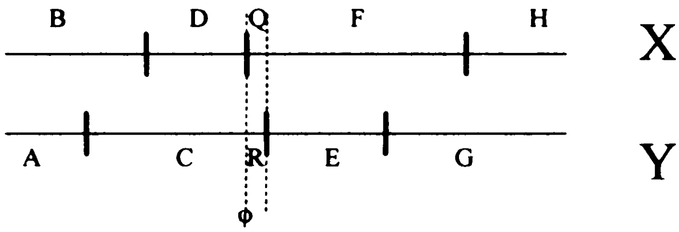

Figure A.7

The *a priori* expected value can be calculated as follows:

$$\begin{aligned}
E[V_1] &= \ldots \\
+E[V &\mid Q \cup F] \Pr[Q \cup F \text{ highest}] \\
+E[V &\mid E] \Pr[E \text{ highest}] \\
+E[V &\mid D] \Pr[D \text{ highest}] \\
+E[V &\mid C \cup R] \Pr[C \cup R \text{ highest}] \\
& +\ldots
\end{aligned}$$

Consider the change from $E[V_1]$ to $E[V_2]$ if every interval threshold $t_i \geq \phi$ is switched between X and Y.

$$\begin{array}{rcl} E[V_2] & = & \ldots \\ +E[V & \mid & F]\Pr[F \text{ highest}] \\ +E[V & \mid & E \cup R]\Pr[\text{"}E \text{ or } R\text{" highest}] \\ +E[V & \mid & D \cup Q]\Pr[\text{"}D \text{ or } Q\text{" highest}] \\ +E[V & \mid & C\,]\Pr[C \text{ highest}] \\ & & +\ldots \end{array}$$

Appendix A.4 shows that if one realisation is higher than F then independence of irrelevant alternatives means there is no change; if both realisations are lower than C then there is no change. Therefore the unexpressed terms will be equal so

$$\begin{aligned} E[V_2] - E[V_1] &= Ff\,(1-g) + (eE + rR)\,(1-h-f) \\ &\quad + (dD + qQ)\,(1-g-e-r) + Cc\,(1-f-h-q-d) \\ &\quad - (Qq + Ff)\,(1-g) - Ee\,(1-h-f-q) \\ &\quad - Dd\,(1-g-e) - (Cc + rR)\,(1-h-f-q-d) \end{aligned} \tag{A.4}$$

Observing in Figure A.7 that $Q = R$, $q = r$, $a+c = b+d$ and $e+g = f+h$ reduces Equation A.4 to Equation A.5.

$$E[V_2] - E[V_1] = qd\,[Q - D] + qe\,[E - Q] > 0 \text{ as } E > Q > D \tag{A.5}$$

This proof can be extended to the case of more projects by holding the expected values constant in the decision stage (reoptimisation will lead to a larger improvement). If any interval strictly contains an interval on another project (and the situation is not degenerate, when a realisation lieing in neighbouring intervals would lead to the same decision for all realisations of other project values) then it is always possible to increase expected value with such a transformation. It is sufficient to examine local intervals because of independence from irrelevant alternatives.

A.6 Optimal Internal Intervals Do Not Weakly Contain Intervals on Other Projects

Continuing the notation of Appendix A.5, consider the change in expected utility from increasing one of the common interval partitions in Figure A.8 by a small amount, so that the interval Q will be grouped with the interval D when communicating to the decision maker rather than grouped with F, as illustrated in Figure A.8. The change is small and made in a direction so the optimal choice given any signal combination remains the same. Note that a switch described in Appendix A.4 may be necessary so that increasing the partition unambiguously increases expected value.

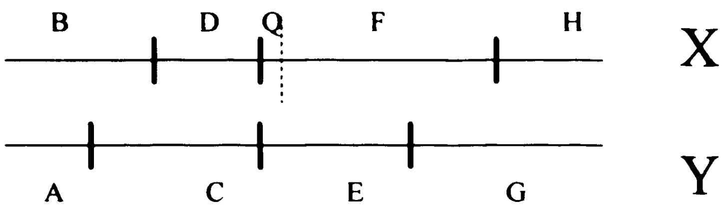

Figure A.8

The expected value can be decomposed as follows:

$$E[V] = E[V \mid X < Q] + E[V \mid Q] + E[V \mid X > Q]$$

As the change is small, $E[V \mid X < Q]$ and $E[V \mid X > Q]$ will be the same before and after Q is included in the signal with interval D (there is no change the order of expected values). The change in expected value depends only on the effect on $E[V \mid Q]$. When Q is grouped with F:

$$E[V_1 \mid Q] = E\left[V \mid F \cup Q \text{ not chosen}\right] \Pr\left[F \cup Q \text{ not chosen}\right] + Q \Pr\left[\ F \cup Q \text{ chosen}\right]$$

After changing Q to be included with D:

$$E[V_2 \mid Q] = E\left[V \mid D \cup Q \text{ not chosen}\right] \Pr\left[D \cup Q \text{ not chosen}\right] + Q \Pr\left[D \cup Q \text{ chosen}\right]$$

Therefore:

$$\begin{aligned}
E[V_2] - E[V_1] &= q\left(E[V_2 \mid Q] - E[V_1 \mid Q]\right) \\
&= qQ\left[\Pr\left[D \cup Q \text{ chosen}\right] - \Pr\left[\ F \cup Q \text{ chosen}\right]\right] \\
&\quad + qE\left[V \mid D \cup Q < \text{some } E\left[V\right] < F \cup Q\right] \Pr\left[V \mid D \cup Q < \text{some } E\left[V\right] < F \cup Q\right] \\
&= qQ\left[(1 - g - e) - (1 - g)\right] + q\left[Ee\right] \\
&= qe\left(E - Q\right)
\end{aligned}$$

As $E > Q$ this is always positive, and therefore increasing one interval by a small amount always improves the solution. This argument extends to the case when there are many projects. Note that the gain equals the expected error, as E is now correctly chosen when before Q was chosen in error.

Appendix B

Endogenous Analogy Classes

B.1 Optimality Based Refinement of the Centipede Game when $N = 2$

Before optimality based refinement of mixed strategy analogy-based expectations equilibria of the Centipede game when $N = 2$, it is useful to observe that the full information expected payoffs are independent of the specific equilibrium. For player 1, the expected true payoff is:

$$\begin{aligned} U_1[p\ ,\ q] &= 1p_1 + 2(1-p_1)(1-q_1)p_2 + 1(1-p_1)(1-q_1)(1-p_2)q_2 \\ &\quad +3(1-p_1)(1-q_1)(1-p_2)(1-q_2) \\ &= p_1 + (1-p_1)(1-q_1)(1+p_2) \\ &= 1 \end{aligned}$$

For 2, the expected true payoff is:

$$\begin{aligned} U_2[p\ ,\ q] &= 1p_1 + 3(1-p_1)q_1 + 2(1-p_1)(1-q_1)p_2 + 4(1-p_1)(1-q_1)(1-p_2)q_2 \\ &\quad +3(1-p_1)(1-q_1)(1-p_2)(1-q_2) \\ &= p_1 + 3(1-p_1)q_1 + 2(1-p_1)(1-q_1)(2-p_2) \\ &= 2 \end{aligned}$$

The optimality based refinement measures were specified in Definition 10.2 as $\frac{U_1[p'\ ,\ q]-U_1[p\ ,\ q]}{U_1[p\ ,\ q]-A_{1,2}}$ for player 1 and $\frac{U_2[p\ ,\ q']-U_2[p\ ,\ q]}{U_2[p\ ,\ q]-B_{1,1}}$ for player 2. Following Proposition 10.1 some pure strategy p' and q' must be optimal for each player. The payoffs from each pure strategy can be calculated for player 1:

	Pure Strategy p'	Expected Payoff from p', $U_1[p'\ ,\ q]$	$\frac{U_1[p'\ ,\ q]-U_1[p\ ,\ q]}{U_1[p\ ,\ q]-A_{1,2}}$
(1)	*Take, Take*	1	$\frac{1-1}{1}$
(2)	*Pass, Take*	$2(1-q_1)$	$\frac{2(1-q_1)-1}{1}$
(3)	*Pass, Take*	$(1-q_1)[q_2+3(1-q_2)]$	$\frac{(1-q_1)[q_2+3(1-q_2)]-1}{1}$

For player 2:

	Pure Strategy q'	Expected Payoff from $q', U_1\,[p\ ,\ q']$	$\frac{U_2[p\ ,\ q']-U_2[p\ ,\ q]}{U_2[p\ ,\ q]-B_{1,\ 1}}$
(4)	*Take, Take*	$p_1 + 3(1-p_1)$	$\frac{p_1+3(1-p_1)-2}{2-1}$
(5)	*Pass, Take*	$p_1 + 2p_2(1-p_1) + 4(1-p_1)(1-p_2)$	$\frac{p_1+2p_2(1-p_1)+4(1-p_1)(1-p_2)-2}{2-1}$
(6)	*Pass, Pass*	$p_1 + 2p_2(1-p_1) + 3(1-p_1)(1-p_2)$	$\frac{a+2p_2(1-p_1)+3(1-p_1)(1-p_2)-2}{2-1}$

Condition 6 will never bind as (*Pass, Take*) always has a higher payoff for player 2, which is another way of expressing that for optimality $q_2 = 1$. Indirectly this rules out condition 3 ever binding, as the payoff from (*Pass, Take*) will be greater for player 1. Finally as $q_2 \leq \widehat{q}$, condition 1 will never bind alone. This leaves the following conditions:

$$\begin{array}{ll} (2) & t_1 = 1 - 2q_1 \\ (4) & t_2 = 1 - 2p_1 \\ (5) & t_2 = 2 - 3p_1 + 2p_2\,(p_1 - 1) \end{array}$$

These can be expressed as restrictions on p_1 using the consistency requirements shown in Table 9.2 that $q_1 = 1 - \frac{1}{3(1-p_1)}$ and $p_2 = 2 - 3p_1$, giving the following measures of t_1 and t_2:

$$\begin{aligned} t_1(p\ ,\ q) &= \frac{2}{3\,(1-p_1)} - 1 \\ t_2(p\ ,\ q) &= \begin{cases} 1 - 2p_1 & \text{if } p_1 \leq \frac{1}{2} \\ 7p_1 - 6p_1^2 - 2 & \text{if } p_1 \geq \frac{1}{2} \end{cases} \end{aligned}$$

This is illustrated in Figure 10.6.

B.2 Optimality Based Refinement of the Doubling Dollar Game When $N = 2$

For the Doubling Dollar game when $N = 2$ in a mixed strategy analogy-based expectations equilibrium, as characterised in Table 9.2, player 1 has a true average payoff of $\frac{4}{3} - p_1$ and player 2 has a true average payoff of p_1. As $A_{1,2} = B_{1,1} = 0$ in the Doubling Dollar game, $t_1 = \frac{U_1[p'\ ,\ q]-U_1[p\ ,\ q]}{U_1[p\ ,\ q]-A_{1,2}} = \frac{U_1[p'\ ,\ q]}{U_1[p\ ,\ q]} - 1$ for player 1 and $t_2 = \frac{U_2[p\ ,\ q']-U_2[p\ ,\ q]}{U_2[p\ ,\ q]-B_{1,\ 1}} = \frac{U_2[p\ ,\ q']}{U_2[p\ ,\ q]} - 1$ for player 2. Following Proposition 10.1 some pure strategy p' and q' must be optimal for each player. The payoffs from each pure strategy can be calculated for player 1:

	Pure Strategy p'	Expected Payoff from $p', U_1\,[p'\ ,\ q]$	$\frac{U_1[p'\ ,\ q]}{U_1[p\ ,\ q]} - 1$
(1)	*Take*	1	$\frac{1}{\frac{4}{3}-p_1} - 1$
(2)	*Pass, Take*	$2(1-q_1)$	$\frac{2(1-q_1)}{\frac{4}{3}-p_1} - 1$
(3)	*Pass, Pass*	$4(1-q_1)(1-q_2)$	$\frac{4(1-q_1)(1-q_2)}{\frac{4}{3}-p_1} - 1$

While for player 2:

	Pure Strategy q'	Expected Payoff from $q', U_1[p\ ,\ q']$	$\frac{U_2[p\ ,\ q']}{U_2[p\ ,\ q]} - 1$
(4)	*Take*	$1(1-p_1)$	$\frac{1(1-p_1)}{p_1} - 1$
(5)	*Pass, Take*	$2(1-p_1)(1-p_2)$	$\frac{2(1-p_1)(1-p_2)}{p_1} - 1$
(6)	*Pass, Pass*	0	$\frac{0}{p_1} - 1$

As in Appendix B.1 it is sufficient to examine conditions 2, 3 and 4 as conditions 1, 3 and 6 will never bind. This leaves the following conditions:

$$
\begin{aligned}
&(2) \quad t_1 = \frac{2(1-q_1)}{\frac{4}{3}-p_1} - 1 \\
&(4) \quad t_2 = \frac{1-p_1}{p_1} - 1 \\
&(5) \quad t_2 = \frac{2(1-p_1)(1-p_2)}{p_1} - 1
\end{aligned}
$$

These can be expressed as restrictions on p_1 using the consistency requirements shown in Table 9.2 that $q_1 = 1 - \frac{1}{3(1-p_1)}$ and $p_2 = 2 - 3p_1$, giving the following measures of t_1 and t_2:

$$
\begin{aligned}
t_1(p\ ,\ q) &= \frac{2}{(4-3p_1)(1-p_1)} - 1 \\
t_2(p\ ,\ q) &= \begin{cases} \frac{1-p_1}{p_1} - 1 & \text{if } p_1 \leq \frac{1}{2} \\ \frac{2(1-p_1)(3p_1-1)}{p_1} - 1 & \text{if } p_1 \geq \frac{1}{2} \end{cases}
\end{aligned}
$$

This is illustrated in Figure 10.7.

B.3 Some Pure Strategy is at Least Weakly Optimal

Consider the game illustrated in Figure B.1, where $A_i, B_i, ..., E_i$

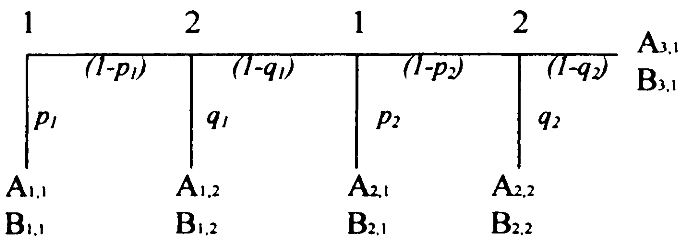

Figure B.1

An expression for the "true" expected payoff for player 1 can be written:

$$
\begin{aligned}
E[\pi_1] &= p_1 A_{1,1} + (1-p_1)\, q_1 A_{1,2} + (1-p_1)(1-q_1)\, p_2 A_{2,1} \\
&\quad + (1-p_1)(1-q_1)(1-p_2)\, q_2 A_{2,2} + (1-p_1)(1-q_1)(1-p_2)(1-q_2)\, A_{3,1} \\
&= p_1 [A_{1,1}] + (1-p_1)\, p_2 [q_1 A_{1,2} + (1-q_1) A_{2,1}] \\
&\quad + (1-p_1)(1-p_2)[q_1 A_{1,2} + (1-q_1)\, q_2 A_{2,2} + (1-q_1)(1-q_2) A_{3,1}] \\
&= p_1 E[\pi_1 \mid p_1 = 1] \\
&\quad + (1-p_1)\, p_2 E[\pi_1 \mid p_1 = 0,\ p_2 = 1] \\
&\quad + (1-p_1)(1-p_2)[\pi_1 \mid p_1 = 0,\ p_2 = 0]
\end{aligned}
$$

Therefore payoffs from mixing will be a convex combination of the payoff from using playing pure strategies. The same is true for player 2 by symmetry.

B.4 Derivation of Equation 10.1 in Proposition 10.6

If $p_n = \tilde{p}\ \forall\ n$ the expected number of nodes $X_{n,2}$ observed is:

$$
\begin{aligned}
\sum_{X_{n,2}} \Pr_{p,q}(X_{n,2}) &= \sum_{n=1}^{g} \left[(1-\tilde{q})^{n-1} \prod_{i=1}^{i=n} (1-p_i) \right] \\
&\quad + \left[(1-\tilde{q})^{g-1} (1-q_g) \left(\prod_{i=1}^{i=g} (1-p_i) \right) \right] \sum_{n=g+1}^{N} \left[(1-\tilde{q})^{n-g-1} \prod_{i=g+1}^{n-1} (1-p_n) \right] \\
&= \sum_{n=1}^{g} \left[(1-\tilde{q})^{n-1} (1-\tilde{p})^{n} \right] \\
&\quad + \left[(1-\tilde{q})^{g-1} (1-q_g)(1-\tilde{p})^{g} \right] \sum_{n=g+1}^{N} \left[(1-\tilde{q})^{n-g-1} (1-\tilde{p})^{n-g-1} \right] \\
&= (1-\tilde{p}) \left[\frac{1-(1-\tilde{q})^{g}(1-\tilde{p})^{g}}{1-(1-\tilde{q})(1-\tilde{p})} \right] \\
&\quad + \left[\frac{(1-\tilde{q})^{g-1}(1-\tilde{q})(1-\tilde{p})^{g}}{1-(1-\tilde{q})^{N-g}(1-\tilde{p})^{N-g}} \right] \left[\frac{1-(1-\tilde{q})^{N-g}(1-\tilde{p})^{N-g}}{1-(1-\tilde{q})(1-\tilde{p})} \right] \\
&= \frac{1-\tilde{p}+\tilde{p}\,[(1-\tilde{q})(1-\tilde{p})]^{g}}{\tilde{q}+\tilde{p}-\widetilde{qp}}
\end{aligned}
$$

This expression is decreasing in g, so minimising g maximises the number of times player 2 is expected to move in equilibrium.

Appendix C

Common Value Multi-Unit Auctions

C.1 Solution for Asymmetric First-price, Sealed-bid Auction

Proposition 16.1 derived the first order conditions for the discriminatory auction in Equation 16.1 as:

$$\begin{aligned} b_i^1 &: \left[\frac{1}{2}\left[s_i + \phi_j^2\left(b_i^1\right)\right] - b_i^1\right] \phi_j^{2\prime}\left(b_i^1\right) = \phi_j^2\left(b_i^1\right) \\ b_j^2 &: \left[\frac{\alpha}{2}\left[s_j + \phi_i^1\left(b_j^2\right)\right] - b_j^2\right] \phi_i^{1\prime}\left(b_j^2\right) = \phi_i^1\left(b_j^2\right) \end{aligned}$$

Appendix C.1 derives a closed form solution to these first order conditions. It focuses on an asymmetric first-price auction for a single unit, where bidders have independent signals which are uniformly distributed over $[0\ ,\ 1]$, and bidder i has a valuation of $\frac{s_i+s_j}{2}$ while bidder j has a valuation of $\alpha\frac{s_i+s_j}{2}$. The superscripts are dropped for simplicity. The differential equations can be solved analytically as they can be rewritten as in Equation C.1.

$$\begin{aligned} \frac{1}{2}\left[\phi_i(b) + \phi_j(b)\right] \phi_j^{'}(b) &= \phi_j(b) + b\phi_j^{'}(b) \\ \frac{1}{2}\left[\phi_i(b) + \phi_j(b)\right] \phi_i^{'}(b) &= \frac{\phi_i(b) + b\phi_i^{'}(b)}{\alpha} \end{aligned} \tag{C.1}$$

These sum to give

$$\frac{1}{2}\left[\phi_i(b) + \phi_j(b)\right] \phi_j^{'}(b) + \frac{1}{2}\left[\phi_i(b) + \phi_j(b)\right] \phi_i^{'}(b) = \phi_j(b) + b\phi_j^{'}(b) + \frac{\phi_i(b) + b\phi_i^{'}(b)}{\alpha}$$

Integrating both sides (the constant is 0 as $\phi_i(0) = \phi_j(0) = 0$) gives

$$\begin{aligned} \frac{1}{4}\left(\phi_i(b) + \phi_j(b)\right)^2 &= b\phi_j(b) + \frac{b\phi_i(b)}{\alpha} \\ &= b\left(\phi_j(b) + \frac{\phi_i(b)}{\alpha}\right) \end{aligned} \tag{C.2}$$

The upper bound on the support of the bids must be the same for both bidders:

$$\frac{1}{4}(1+1)^2 = \bar{b}\left(1+\frac{1}{\alpha}\right) \Rightarrow \bar{b} = \frac{\alpha}{1+\alpha} \tag{C.3}$$

Therefore Equation C.2 can be rewritten as C.4 or C.5.

$$\phi_i(b)^2 + \phi_i(b)\left(2\phi_j(b) - \frac{4b}{\alpha}\right) + \left(\phi_j(b)^2 - 4b\phi_j(b)\right) = 0 \tag{C.4}$$

$$\phi_j(b)^2 + \phi_j(b)\left(2\phi_i(b) - 4b\right) + \left(\phi_i(b)^2 - 4b\frac{\phi_i(b)}{\alpha}\right) = 0 \tag{C.5}$$

These have positive roots given in Equations C.6 and C.7.

$$\phi_i(b) = \frac{2b}{\alpha} - \phi_j(b) + 2\sqrt{b\phi_j(b)\left(\frac{\alpha-1}{\alpha}\right) + \frac{b^2}{\alpha^2}} \tag{C.6}$$

$$\phi_j(b) = 2b - \phi_i(b) + 2\sqrt{b\phi_i(b)\left(\frac{1-\alpha}{\alpha}\right) + b^2} \tag{C.7}$$

These can be substituted into the first order conditions to give two differential equations C.8 and C.9 in terms of only one of the variables:

$$\left[b\left(\frac{1-\alpha}{\alpha}\right) + \sqrt{b\phi_j(b)\left(\frac{\alpha-1}{\alpha}\right) + \frac{b^2}{\alpha^2}}\right]\phi_j'(b) = \phi_j(b) \tag{C.8}$$

$$\left[b\left(\frac{\alpha-1}{\alpha}\right) + \sqrt{b\phi_i(b)\left(\frac{1-\alpha}{\alpha}\right) + b^2}\right]\phi_i'(b) = \frac{\phi_i(b)}{\alpha} \tag{C.9}$$

Note that if $\alpha = 1$ then this reduces to $b\phi_i'(b) = \phi_i(b)$ for both $i = 1$, 2 which integrates to give $\phi_i(b) = kb$, and a linear bidding function is sufficient to ensure a general optimum. Solving first for bidder i, with a valuation $\frac{s_i+s_j}{2}$, Equation C.8 can be simplified using a change of variables of the form $\phi_i(b) = \frac{\alpha}{1-\alpha}b\left[\xi_i(b)^2 - 1\right] \Rightarrow \phi_i'(b) = \frac{\alpha}{1-\alpha}\left[\xi_i(b)^2 - 1 + 2\xi_i(b)b\xi_i'(b)\right]$ for $\alpha \neq 1$.

$$\frac{2\xi_i(b)\left[\alpha - 1 + \alpha\xi_i(b)\right]}{\left(2-\alpha-\alpha\xi_i(b)\right)\left[\xi_i(b)^2-1\right]}d\xi_i = \frac{db}{b}$$

Then expanding by partial fractions and integrating gives

$$\frac{1}{(1-\alpha)2}\int \frac{(2-\alpha)\alpha}{2-\alpha-\alpha\xi_i(b)} + \frac{2\alpha-1}{\xi_i(b)-1} - \frac{1-\alpha}{\xi_i(b)+1}d\xi_i = \int\frac{db}{b} \tag{C.10}$$

$$\frac{\left(\xi_i(b)-1\right)^{(2\alpha-1)}}{\left(2-\alpha-\alpha\xi_i(b)\right)^{(2-\alpha)}\left(\xi_i(b)+1\right)^{(1-\alpha)}} = Ab^{2(1-\alpha)} \tag{C.11}$$

Substituting $\xi_i(b) = \sqrt{\frac{1-\alpha}{\alpha b}\phi_i(b) + 1}$ and solving for A[1] using the condition on $\bar{b}$

[1] $A = \frac{(1-\alpha)^{3(\alpha-1)}(1+\alpha)^{(1-\alpha)}}{\alpha^{(2\alpha-2)}}$

given in Equation C.3 gives an implicit function as an analytical solution for bidder i

$$\frac{\left[-\alpha+\sqrt{\frac{\alpha(1-\alpha)}{b}\phi_i(b)+\alpha^2}\right]^{\alpha}}{\left[2-\alpha-\sqrt{\frac{\alpha(1-\alpha)}{b}\phi_i(b)+\alpha^2}\right]^{(2-\alpha)}}=\left[b\phi_i(b)\left(\frac{1+\alpha}{\alpha(1-\alpha)^2}\right)\right]^{(1-\alpha)} \tag{C.12}$$

Similarly Equation C.9 for a bidder j with a valuation $\alpha\frac{(s_i+s_j)}{2}$ can be simplified using a change of variables of the form $\phi_j(b)=\frac{\alpha}{1-\alpha}b\left[\frac{1}{\alpha^2}-\xi_j(b)^2\right]\Rightarrow\phi_j'(b)$ $=\frac{\alpha}{1-\alpha}\left[\frac{1}{\alpha^2}-2b\xi_j(b)\xi_j'(b)-\xi_j(b)^2\right]$ for $\alpha\neq 1$.

$$\frac{2\xi_j(b)\left[\frac{1}{\alpha}-1+\xi_j(b)\right]}{\left(2-\frac{1}{\alpha}-\xi_j(b)\right)\left[\xi_j(b)^2-\frac{1}{\alpha^2}\right]}d\xi_j=\frac{db}{b} \tag{C.13}$$

Then expanding by partial fractions and integrating gives

$$\int\frac{2\alpha-1}{\xi_j(b)+\frac{1}{\alpha}-2}+\frac{2-\alpha}{\frac{1}{\alpha}-\xi_j(b)}-\frac{1-\alpha}{\frac{1}{\alpha}+\xi_j(b)} = \int 2(1-\alpha)\frac{db}{b}$$

$$\frac{\left[\xi_j(b)+\frac{1}{\alpha}-2\right]^{(2\alpha-1)}}{\left[\frac{1}{\alpha}-\xi_j(b)\right]\left[\frac{1}{\alpha^2}-\xi_j(b)^2\right]^{(\alpha-1)}} = Ab^{2(1-\alpha)}$$

Substituting $\xi_j(b)=\sqrt{\frac{1}{\alpha^2}-\phi_j(b)\frac{1-\alpha}{\alpha b}}$ and solving for A using the condition on $\bar{b}$ given in Equation C.3 gives an implicit function as an analytical solution for bidder j:

$$\frac{\left[\sqrt{1-\phi_j(b)\frac{\alpha(1-\alpha)}{b}}+1-2\alpha\right]^{(2\alpha-1)}}{1-\sqrt{1-\phi_j(b)\frac{(1-\alpha)\alpha}{b}}}=\left[b\phi_j(b)\left(\frac{1+\alpha}{\alpha(1-\alpha)^2}\right)\right]^{(1-\alpha)} \tag{C.14}$$

C.2 Highest Losing Bid Uniform Price Auction

In this uniform price auction the same price is paid for every unit won, that of the highest losing bid. As $b_j^1 \geq b_j^2\ \forall\ s$ if bidder i wins two units he will pay a price of $2b_j^1$. If each bidder wins one unit then they will both pay $Max\left(b_i^2\ ,\ b_j^2\right)$. Bidders i will not know j's bids in advance, but can form expectations of the price they will pay conditional on winning either one or two units. The probabilities of winning one unit and paying b_i^2 or b_j^2 can be calculated as follows:

The probability of bidder i winning one unit only and paying i's second bid $(b_i^2 > b_j^2)$:	$\Pr\left[b_j^1 > b_i^2 > b_j^2\right]$ $= \Pr\left[\phi_j^2\left(b_i^2\right) > s_j > \phi_j^1\left(b_i^2\right)\right]$ $= \phi_j^2\left(b_i^2\right) - \phi_j^1\left(b_i^2\right)$ w
The probability of bidder i winning one unit only and paying j's second bid $(b_j^2 > b_i^2)$:	$\Pr\left[b_i^1 > b_j^2 > b_i^2\right]$ $= \Pr\left[\phi_j^2\left(b_i^1\right) > s_j > \phi_j^2\left(b_i^2\right)\right]$ $= \phi_j^2\left(b_i^1\right) - \phi_j^2\left(b_i^2\right)$

Expected price paid conditional on winning one unit:

$$E\left[p \mid win1\right] = \frac{\varphi_j^2\left(b_i^2\right)}{\varphi_j^2\left(b_i^1\right) - \varphi_j^1\left(b_i^2\right)}$$

Expected total price paid conditional on winning two units:

$$E\left[2b_j^1 \mid win2\right] = \frac{2}{\varphi_j^1\left(b_i^2\right)} \int_0^{\varphi_j^1\left(b_i^2\right)} b_j^1(x)dx$$

These expected payments can be substituted into the expression for expec plus in Equation 15.1.

$$\begin{aligned}
U_i &= E\left[\frac{s_i + s_j}{2} - \tilde{b} \mid win1\right] \Pr\left(win1\right) \\
&\quad + E\left[\frac{1+\alpha}{2}\left(s_i + s_j\right) - 2b_j^1 \mid win2\right] \Pr(win2) \\
&= \frac{1}{4}\left[2s_i + \varphi_j^2\left(b_i^1\right) + \varphi_j^1\left(b_i^2\right)\right]\left[\varphi_j^2\left(b_i^1\right) - \varphi_j^1\left(b_i^2\right)\right] \\
&\quad - \left[\varphi_j^2\left(b_i^2\right) - \varphi_j^1\left(b_i^2\right)\right] b_i^2 - \int_{\varphi_j^2\left(b_i^2\right)}^{\varphi_j^2\left(b_i^1\right)} b_j^2(x)dx \\
&\quad + \left[\frac{1+\alpha}{2}\left[s_i + \frac{\varphi_j^1\left(b_i^2\right)}{2}\right]\varphi_j^1\left(b_i^2\right) - 2\int_0^{\varphi_j^1\left(b_i^2\right)} b_j^1(x)dx\right]
\end{aligned}$$

Differentiating Equation C.17 with respect to b_i^1 and b_i^2 gives the following tions for bidder i to maximise expected surplus:

$$\begin{aligned}
\frac{\partial U_i}{\partial b_i^1} &= \left[\frac{1}{2}\left[s_i + \varphi_j^2\left(b_i^1\right)\right] - b_i^1\right]\varphi_j^{2\prime}\left(b_i^1\right) \\
\frac{\partial U_i}{\partial b_i^2} &= \left[\frac{\alpha}{2}\left[s_i + \varphi_j^1\left(b_i^2\right)\right] - b_i^2\right]\varphi_j^{1\prime}\left(b_i^2\right) + \varphi_j^1\left(b_i^2\right) - \varphi_j^2\left(b_i^2\right)
\end{aligned}$$